CIE IGCSE Additional Maths

Your notes

Differentiation

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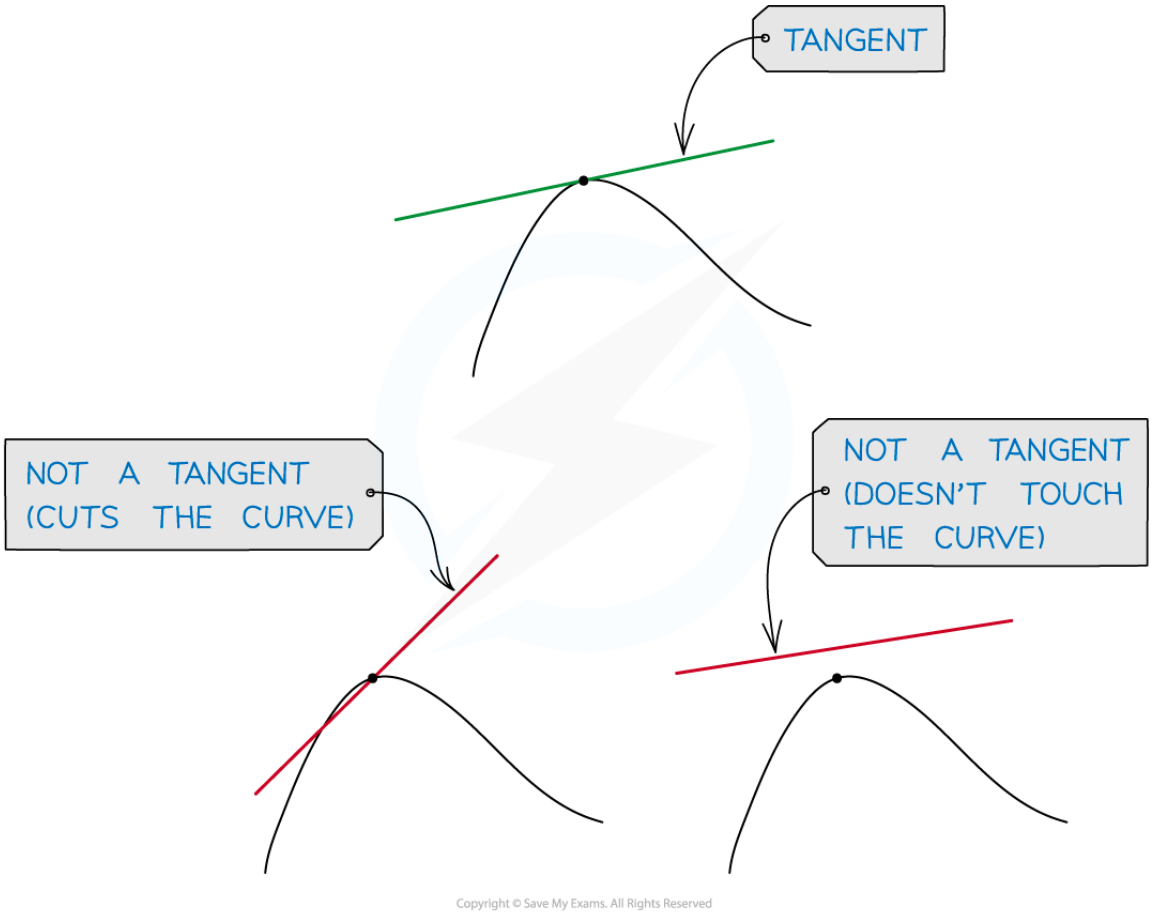
Introduction to Differentiation

Your notes

Definition of Gradient

What is the gradient of a curve?

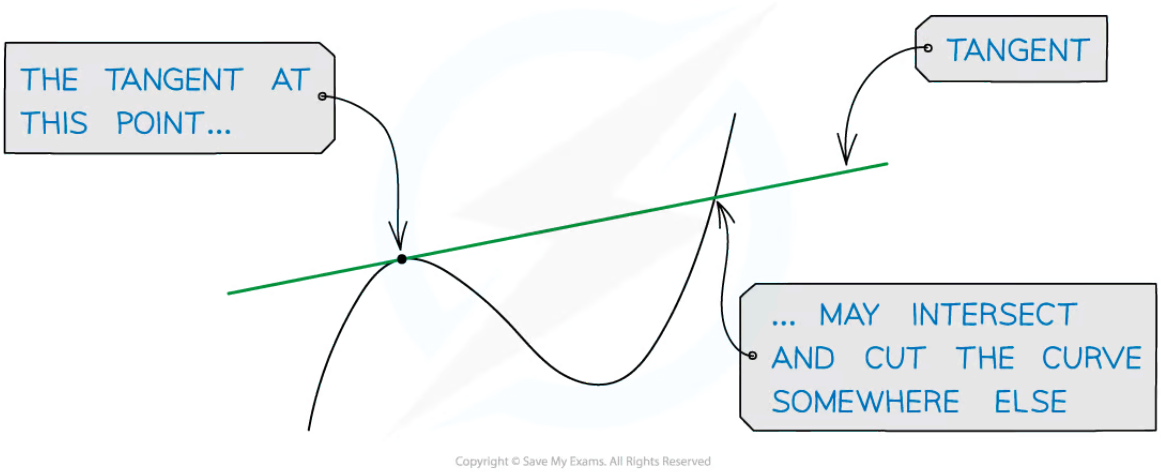
- At a given point the **gradient** of a curve is defined as the gradient of the **tangent** to the curve at that point
- A **tangent** to a curve is a line that just **touches** the curve at one point but doesn't cut the curve at that point



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- A tangent *may* cut the curve somewhere else on the curve

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- It is only possible to draw **one** tangent to a curve at any given point
- Note that unlike the gradient of a straight line, the gradient of a curve is constantly changing

Examiner Tip

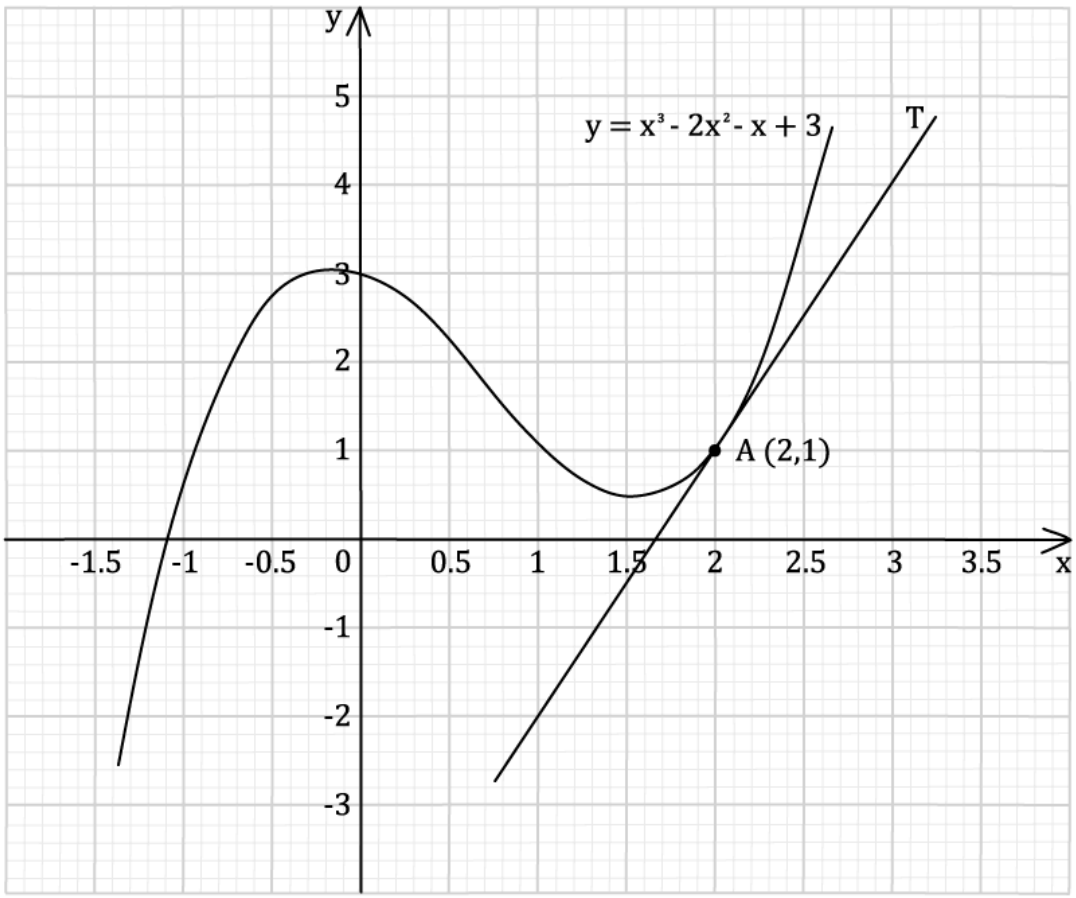
- If a question asks for the "rate of change of ..." then it is asking for the "gradient"

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Worked example

The diagram shows the curve with equation $y = x^3 - 2x^2 - x + 3$. The tangent, T, to the curve at the point $A(2, 1)$ is also shown.

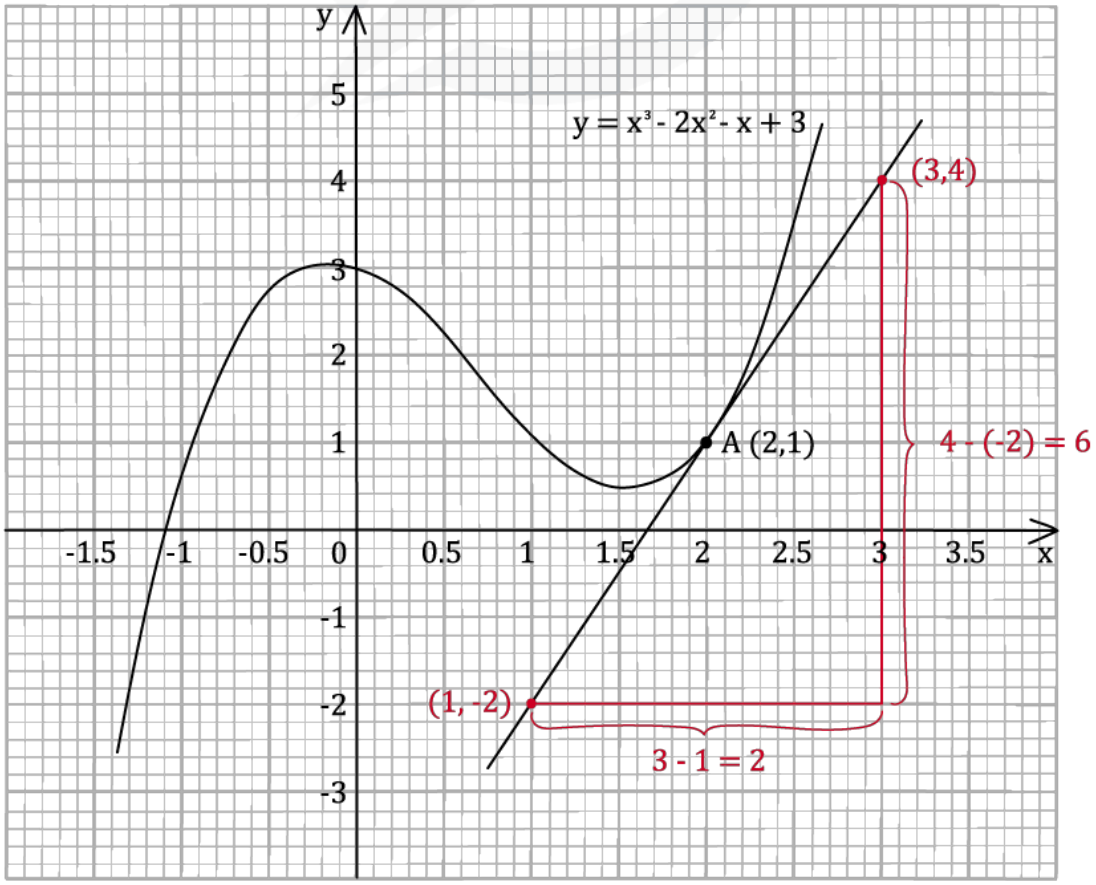


Using the diagram, calculate the gradient of the curve at A.

The gradient of the curve at the point *A* is the same as the gradient of the tangent *T*.
Calculate the gradient of the line.

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$$\frac{4-(-2)}{3-1}$$

The gradient is 3

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Definition of Derivatives

What is a derivative?

Your notes

- **Calculus** is about **rates** of **change**
  - the way a car's position on a road changes is its speed (velocity)
  - the way the car's speed changes is its acceleration
- The **gradient** (rate of change) of a (non-linear) **function** varies with x
- The **derivative** of a function is a function that relates the **gradient** to the value of x
  - For example, the derivative of $y = x^2$ is $2x$
    - This means that when $x = 1$, the gradient of $y = x^2$ is $2(1) = 2$
    - And when $x = 5$, the gradient of $y = x^2$ is $2(5) = 10$
- The derivative is also called the **gradient function**

Worked example

The derivative of $y = x^3 - 2x^2 - x + 3$ is $3x^2 - 4x - 1$.

Use the derivative to find the gradient of $y = x^3 - 2x^2 - x + 3$ at the point $A(2, 1)$.

Substitute $x = 2$ into the derivative, $3x^2 - 4x - 1$

$$3(2)^2 - 4(2) - 1 = 12 - 8 - 1$$

gradient = 3

Note that the answer is the same as in the method above

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Differentiating Powers of x

Your notes

What is differentiation?

- **Differentiation** is the process of finding an expression for the **derivative** (**gradient function**) from the equation of a curve
  - The equation of the curve is written $y = \ldots$ and the gradient function is written $\frac{\mathrm{d}y}{\mathrm{d}x} = \ldots$

How do I differentiate powers of *x*?

- Powers of x are **differentiated** according to the following formula:
  - If $y = ax^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = anx^{n-1}$
    - e.g. If $y = 4x^3$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 4 \times 3 \times x^{3-1} = 12x^2$
    - you "bring down the power" then "subtract one from the power"
- Don't forget these **two** special cases:
  - If $y = ax$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = a$
    - e.g. If $y = 6x$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 6$
  - If $y = a$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
    - e.g. If $y = 5$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
  - These allow you to differentiate **linear terms** in x and **constants**
- Functions involving **fractions** with **denominators** in terms of x will need to be rewritten as **negative powers** of x first
  - e.g. If $y = \frac{4}{x}$ then rewrite as $y = 4x^{-1}$ and differentiate

How do I differentiate sums and differences of powers of *x*?

- The formulae for differentiating powers of x work for a **sum** or **difference** of **powers** of x
  - e.g. If $y = 5x^4 + 3x^{-2} + 4$ then
    $\frac{\mathrm{d}y}{\mathrm{d}x} = 5 \times 4x^{4-1} + 3 \times (-2)x^{-2-1} + 0$

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$$\frac{dy}{dx} = 20x^3 - 6x^{-3}$$

- This is sometimes referred to differentiating 'term-by-term'

- **Products** and **quotients (divisions) cannot** be differentiated in this way so they need **expanding/simplifying** first
  - e.g. If $y = (2x - 3)(x^2 - 4)$ then expand to $y = 2x^3 - 3x^2 - 8x + 12$ which **is** a **sum/difference** of powers of x and can then be differentiated

What can I do with derivatives (gradient functions)?

- The **derivative** can be used to find the **gradient** of a **function** at any point
  - The gradient of a function at a point is equal to the **gradient** of the **tangent** to the curve at that point
  - A question may refer to the gradient of the tangent

Examiner Tip

- Don't try to do too many steps in your head; write the expression in a format that you can differentiate before you actually differentiate it
  - e.g. $y = \frac{1}{x^4} + \frac{2}{x^3}$ can be rewritten as $y = x^{-4} + 2x^{-3}$ which is then far easier to differentiate

Your notes

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Your notes

Worked example

Find the derivative of

(a) $y = 5x^3 + 2x + \frac{3}{x^2} + 8$

Rewrite the $\frac{3}{x^2}$ term

$$y = 5x^3 + 2x + 3x^{-2} + 8$$

Apply the rule for differentiating powers ($y = ax^n$, $\frac{dy}{dx} = anx^{n-1}$) and apply the special cases for the terms $2x$ and 8 ($y = ax$, $\frac{dy}{dx} = a$ and $y = a$, $\frac{dy}{dx} = 0$)

$$\frac{dy}{dx} = 15x^2 + 2 - 6x^{-3}$$

Unless a question specifies there is not usually a need to rewrite/simplify the answer

$$\frac{dy}{dx} = 15x^2 + 2 - \frac{6}{x^3}$$

(b) $y = (2x + 3)^2$

This is a product of two (equal) brackets so cannot be differentiated directly
Expand the brackets to get an expression in powers of x
Take time to get the expansion correct, writing stages out in full if necessary

$$y = (2x + 3)(2x + 3)$$
$$y = 4x^2 + 6x + 6x + 9$$
$$y = 4x^2 + 12x + 9$$

Differentiate 'term-by-term', looking out for those special cases

$$\frac{dy}{dx} = 8x + 12$$

There is a factor of 4 but there is no demand to factorise the final answer in the question

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$$\frac{dy}{dx} = 8x + 12$$

Your notes

(c) $y = \dfrac{8x^6 - x^3}{2x^4}$

This is a quotient so cannot be differentiated directly
Spot the single denominator which means we can split the fraction by the two terms on the numerator

$$y = \frac{8x^6}{2x^4} - \frac{x^3}{2x^4}$$

Simplify using the laws of indices

$$y = 4x^{6-4} - \frac{1}{2}x^{3-4}$$

$$y = 4x^2 - \frac{1}{2}x^{-1}$$

Each term is now a power of x, so differentiate 'term-by-term'

$$\frac{dy}{dx} = 8x + \frac{1}{2}x^{-2}$$

There is demand to simplify or write the answer in a particular form

$$\frac{dy}{dx} = 8x + \frac{1}{2}x^{-2}$$

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Differentiating Special Functions

Your notes

Differentiating Trig Functions

How do I differentiate trig functions?

- For calculus with trigonometric functions, angles must be measured in **radians**
- The derivative of $y = \sin x$ is $\frac{dy}{dx} = \cos x$
- The derivative of $y = \cos x$ is $\frac{dy}{dx} = -\sin x$
- The derivative of $y = \tan x$ is $\frac{dy}{dx} = \sec^2 x$
- The following two sets of relationships can be derived using the **chain rule**, but are useful to know!
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = \sin(ax + b)$ is $\frac{dy}{dx} = a\cos(ax + b)$
  - the derivative of $y = \cos(ax + b)$ is $\frac{dy}{dx} = -a\sin(ax + b)$
  - the derivative of $y = \tan(ax + b)$ is $\frac{dy}{dx} = a\sec^2(ax + b)$
- For the **general** function $f(x)$,
  - the derivative of $y = \sin(f(x))$ is $\frac{dy}{dx} = f'(x)\cos(f(x))$
  - the derivative of $y = \cos(f(x))$ is $\frac{dy}{dx} = -f'(x)\sin(f(x))$
  - the derivative of $y = \tan(f(x))$ is $\frac{dy}{dx} = f'(x)\sec^2(f(x))$

Examiner Tip

- Remember that these rules only work in radians!

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Your notes

Worked example

a) Find $f'(x)$ for the functions

i. $f(x) = \sin x$

ii. $f(x) = \cos 2x$

ii. $f(x) = 3\sin 4x - \cos(2x - 3)$

i.

$$f'(x) = \cos x$$

ii. Use the chain rule or remember that when $y = \cos(ax + b)$, then

$$\frac{dy}{dx} = -a\sin(ax + b)$$

$$f'(x) = -2\sin(2x)$$

iii. Differentiate 'term by term'

$$f'(x) = 3(4\cos 4x) - (-2\sin(2x - 3))$$

$$f'(x) = 12\cos 4x + 2\sin(2x - 3)$$

b) Find the gradient of the tangent to the curve $y = \sin\left(2x + \frac{\pi}{6}\right)$ at the point where $x = \frac{\pi}{8}$.

Gradient of tangent is equal to gradient of curve. To find the gradient, differentiate...

$$\frac{dy}{dx} = 2\cos\left(2x + \frac{\pi}{6}\right)$$

... and substitute $x = \frac{\pi}{8}$ into the derivative

$$\frac{dy}{dx} = 2\cos\left(2\left(\frac{\pi}{8}\right) + \frac{\pi}{6}\right)$$

Ensure that your calculator is in radians

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$$= \frac{\sqrt{6} - \sqrt{2}}{2} = 0.517638090...$$

Answer = $\frac{\sqrt{6} - \sqrt{2}}{2}$, or 0.518 to 3 significant figures

Your notes

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Differentiating e^x & lnx

How do I differentiate exponentials and logarithms?

Your notes

- The derivative of $y = e^x$ is $\frac{dy}{dx} = e^x$ where $x \in \mathbb{R}$
- The derivative of $y = \ln x$ is $\frac{dy}{dx} = \frac{1}{x}$ where $x > 0$
- In addition, the results below can all be found using the **chain rule** but are worthwhile knowing, to save time in an exam
- For the **linear** function $ax + b$, where a and b are constants,
  - the derivative of $y = e^{(ax+b)}$ is $\frac{dy}{dx} = ae^{(ax+b)}$
  - the derivative of $y = \ln(ax + b)$ is $\frac{dy}{dx} = \frac{a}{(ax+b)}$
    - in the special case $b = 0$, $\frac{dy}{dx} = \frac{1}{x}$ (a's cancel)
- For the **general** function $f(x)$,
  - the derivative of $y = e^{f(x)}$ is $\frac{dy}{dx} = f'(x)e^{f(x)}$
  - the derivative of $y = \ln(f(x))$ is $\frac{dy}{dx} = \frac{f'(x)}{f(x)}$

Examiner Tip

- Remember to avoid the common mistakes:
  - the derivative of $\ln kx$ with respect to x is $\frac{1}{x}$, NOT $\frac{k}{x}$!
  - the derivative of e^{kx} with respect to x is ke^{kx}, NOT kxe^{kx-1} !

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Worked example

A curve has the equation $y = e^{-3x} + 2\ln x$.

Find the gradient of the curve at the point where $x = 2$ giving your answer in the form $a + be^c$, where a, b and c are integers to be found.

Differentiate each term separately.

$$\frac{dy}{dx} = -3e^{-3x} + 2\left(\frac{1}{x}\right)$$

Substitute $x = 2$ into the derivative.

$$\frac{dy}{dx} = -3e^{-3(2)} + 2\left(\frac{1}{2}\right)$$

Rearrange to the required form.

$$\frac{dy}{dx} = -3e^{-6} + 1 = 1 - 3e^{-6}$$

Do not use your calculator to evaluate this as the question asks for the answer given in this form.

$1 - 3e^{-6}$

Your notes

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Chain Rule


Your notes

Chain Rule

What is the chain rule?

- The **chain rule** states if y is a function of u and u is a function of x then

$$y = \mathrm{f}(u(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}u} \times \frac{\mathrm{d}u}{\mathrm{d}x}$$

- In **function notation** this could be written

$$y = \mathrm{f}(\mathrm{g}(x))$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(\mathrm{g}(x))\mathrm{g}'(x)$$

How do I know when to use the chain rule?

- The chain rule is used when we are trying to differentiate **composite functions**
  - "function of a function"
  - these can be identified as the variable (usually x) does not 'appear alone'
    - $\sin x$ - **not** a composite function, x 'appears alone'
    - $\sin(3x+2)$ **is** a **composite function;** x is tripled and has 2 added to it before the sine function is applied

How do I use the chain rule?

STEP 1

Identify the two functions

Rewrite y as a function of u; $y = \mathrm{f}(u)$

Write u as a function of x; $u = \mathrm{g}(x)$

STEP 2

Differentiate y with respect to u to get $\frac{\mathrm{d}y}{\mathrm{d}u}$

Differentiate u with respect to x to get $\frac{\mathrm{d}u}{\mathrm{d}x}$

STEP 3

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Obtain $\frac{dy}{dx}$ by applying the formula $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$ and substitute u back in for $g(x)$

Your notes

- In trickier problems **chain rule** may have to be applied **more than once**

How do I differentiate $(ax + b)^n$?

- For $n = 2$ you will most likely **expand the brackets** and differentiate each term separately
- If $n > 2$ this becomes **time-consuming** and if n is not a positive integer we need a different method completely
- The chain rule allows us to use **substitution** to differentiate any function in the form $y = (ax + b)^n$
  - Let $u = ax + b$, then $y = u^n$
  - **Differentiate** both parts **separately**
    - $\frac{du}{dx} = a$ and $\frac{dy}{du} = nu^{n-1}$
  - Put both parts into the **chain rule**
    - $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = a \times nu^{n-1} = anu^{n-1}$
  - **Substitute** $u = ax + b$ back into your answer
    - $\frac{dy}{dx} = an(ax + b)^{n-1}$

How do I differentiate √(ax+b)?

- The chain rule allows us to use **substitution** to differentiate any function in the form $y = \sqrt{ax + b}$
- Rewrite $\sqrt{ax + b} = (ax + b)^{\frac{1}{2}}$
  - Let $u = ax + b$, then $y = u^{½}$
  - **Differentiate** both parts **separately**
    - $\frac{du}{dx} = a$ and $\frac{dy}{du} = \frac{1}{2}u^{-\frac{1}{2}}$
  - Put both parts into the **chain rule**
    - $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx} = a \times \frac{1}{2}u^{-\frac{1}{2}} = \frac{a}{2}u^{-\frac{1}{2}}$
  - **Substitute** $u = ax + b$ back into your answer
    - $\frac{dy}{dx} = \frac{a}{2}(ax + b)^{-\frac{1}{2}} = \frac{a}{2\sqrt{ax + b}}$
- This method can be used for any **fractional power** of any linear or non-linear expression
  - Provided you know how to differentiate the non-linear expression

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Are there any standard results for using chain rule?

Your notes

- The following general results are particularly useful
  - If $y = (\mathrm{f}(x))^n$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = n\mathrm{f}'(x)\mathrm{f}(x)^{n-1}$
  - If $y = \mathrm{e}^{\mathrm{f}(x)}$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)e^{\mathrm{f}(x)}$
  - If $y = \ln(\mathrm{f}(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{f}'(x)}{\mathrm{f}(x)}$
  - If $y = \sin(\mathrm{f}(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)\cos(\mathrm{f}(x))$
  - If $y = \cos(\mathrm{f}(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = -\mathrm{f}'(x)\sin(\mathrm{f}(x))$
  - If $y = \tan(\mathrm{f}(x))$ then $\frac{\mathrm{d}y}{\mathrm{d}x} = \mathrm{f}'(x)\sec^2(\mathrm{f}(x))$

Examiner Tip

- You should aim to be able to spot and carry out the chain rule mentally (rather than use substitution)
  - every time you use it, say it to yourself in your head
    "differentiate the first function ignoring the second, then multiply by the derivative of the second function"

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Your notes

Worked example

a) Find the derivative of $y = (x^2 - 5x + 7)^7$.

STEP 1 Identify the two functions and rewrite

$$y = u^7 \qquad u = x^2 - 5x + 7$$

STEP 2 Find $\frac{dy}{du}$ and $\frac{du}{dx}$

$$\frac{dy}{du} = 7u^6 \qquad \frac{du}{dx} = 2x - 5$$

STEP 3 Apply the chain rule, $\frac{dy}{dx} = \frac{dy}{du} \times \frac{du}{dx}$

$$\frac{dy}{dx} = 7u^6(2x - 5)$$

Substitute u in terms of x back in

$$= 7(x^2 - 5x + 7)^6(2x - 5)$$

$$\frac{dy}{dx} = 7(2x - 5)(x^2 - 5x + 7)^6$$

b) Find the derivative of $y = \sin(e^{2x})$.

(In this solution, we will be applying the mental method discussed in the Exam Tip above)

"... differentiate $\sin\square$, ignore e^{2x}"

$$\frac{dy}{dx} = \cos(e^{2x}) \times \square$$

"... multiply by the derivative of e^{2x}": differentiate e^{2x} using the result "if $y = e^{f(x)}$, then $\frac{dy}{dx} = f'(x)e^{f(x)}$"

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$$\frac{dy}{dx} = \cos(e^{2x}) \times 2e^{2x}$$

$$\frac{dy}{dx} = 2e^{2x}\cos(e^{2x})$$

Your notes

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Product Rule

Your notes

Product Rule

What is the product rule?

- The product rule is a formula that allows you to differentiate a **product of two functions**
- If $y = u \times v$ where u and v are functions of x then the product rule is:

$$\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$$

- In **function notation**, if $\mathbf{f}(x) = \mathbf{g}(x) \times \mathbf{h}(x)$ then the product rule can be written as:

$$\mathrm{f}'(x) = \mathrm{g}(x)\mathrm{h}'(x) + \mathrm{h}(x)\mathrm{g}'(x)$$

- The easiest way to remember the product rule is, for $y = u \times v$ where u and v are functions of x:

$$y' = uv' + vu'$$

How do I know when to use the product rule?

- The **product rule** is used when we are trying to **differentiate** the **product** of **two functions**
  - These can easily be confused with composite functions (see **chain rule)**
    - $\sin(\cos x)$ is a composite function, "sin of cos of x"
    - $\sin x \cos x$ is a product, "sin x times cos x"

How do I use the product rule?

- Make it clear what u, v, u' and v' are
  - arranging them in a square can help
    - opposite diagonals match up

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{\mathrm{d}y}{\mathrm{d}x}$ by applying the product rule formula $\frac{\mathrm{d}y}{\mathrm{d}x} = u\frac{\mathrm{d}v}{\mathrm{d}x} + v\frac{\mathrm{d}u}{\mathrm{d}x}$

Simplify the answer if straightforward to do so or if the question requires a particular form

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Your notes

e.g. $y = xe^x$ ← $u = x,\ v = e^x$

$$\frac{dy}{dx} = (x)(e^x) + (e^x)(1)$$

u · $\frac{dv}{dx}$ · v · $\frac{du}{dx}$

$$\frac{dy}{dx} = xe^x + e^x$$

$f(x) = x^3(x^2 + 7)$

$f'(x) = (x^3)(2x) + (x^2+7)(3x^2)$ ← $g(x) = x^3,\ h(x) = x^2 + 7$

$g(x)$ · $h'(x)$ · $h(x)$ · $g'(x)$

$f'(x) = 5x^4 + 21x^2$

COMPARE $f(x) = x^3(x^2+7)$
$f(x) = x^5 + 7x^3$
$f'(x) = 5x^4 + 21x^2$

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Examiner Tip

- The product rule formula is **not** on the list of formulas page – make sure you know it
- Don't confuse the **product** of two functions with a **composite function:**
  - The **product** of two functions is two functions **multiplied** together
  - A **composite** function is a **function of a function**

e.g. $f(x) = 3x^2$ $g(x) = x - 3$

$f(x)g(x) = 3x^2(x-3) = 3x^3 - 9x^2$ PRODUCT

$fg(x) = f(g(x)) = 3(x-3)^2 = 3x^2 - 18x + 27$ COMPOSITE FUNCTION

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- To differentiate composite functions you need to use the **chain rule**

Your notes

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Worked example

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Your notes

Differentiate with respect to x:

a) $(5\sin x - 7\cos x)e^{3x+2}$

b) $5x\ln 5x$

a) $u = 5\sin x - 7\cos x$ $\quad v = e^{3x+2}$

$\frac{du}{dx} = 5\cos x + 7\sin x$ $\quad \frac{dv}{dx} = 3e^{3x+2}$

DON'T FORGET MINUS SIGN WHEN YOU DIFFERENTIATE cos

USE CHAIN RULE

$\frac{dy}{dx} = u\frac{dv}{dx} + v\frac{du}{dx}$

$\frac{dy}{dx} = (5\sin x - 7\cos x)(3e^{3x+2}) + (e^{3x+2})(5\cos x + 7\sin x)$

$\frac{dy}{dx} = (22\sin x - 16\cos x)e^{3x+2}$

b) $u = 5x$ $\quad v = \ln 5x$

$\frac{du}{dx} = 5$ $\quad \frac{dv}{dx} = \frac{1}{x}$

BE CAREFUL WITH THIS ONE! BECAUSE OF THE CHAIN RULE THE DERIVATIVE OF $\ln(ax)$ IS $\frac{1}{x}$ FOR **ANY** CONSTANT a

$\frac{dy}{dx} = (5x)\left(\frac{1}{x}\right) + (\ln 5x)(5)$

$\frac{dy}{dx} = 5 + 5\ln 5x$

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Your notes

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Quotient Rule

Your notes

Quotient Rule

What is the quotient rule?

- The quotient rule is a formula that allows you to differentiate a **quotient of two functions**
  - i.e. one function divided by another
- If $y = \frac{u}{v}$ where u and v are functions of x then the quotient rule is:

$$\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$$

- In **function notation**, if $f(x) = \frac{g(x)}{h(x)}$ then the quotient rule can be written as:

$$f'(x) = \frac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$$

- As with the product rule, '**dash notation**' may be used to make remembering it easier

$$y = \frac{u}{v}$$

$$y' = \frac{vu' - uv'}{v^2}$$

- Final answers should match the notation used throughout the question

How do I know when to use the quotient rule?

- The **quotient rule** is used when trying to differentiate a fraction where **both** the **numerator** and **denominator** are **functions** of x
  - if the **numerator** is a **constant**, **negative powers** can be used
  - if the **denominator** is a **constant**, treat it as a **factor** of the expression\

How do I use the quotient rule?

- Make it clear what u, v, u' and v' are

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- arranging them in a square can help
  - opposite diagonals match up (like they do for product rule)


Your notes

STEP 1

Identify the two functions, u and v

Differentiate both u and v with respect to x to find u' and v'

STEP 2

Obtain $\frac{dy}{dx}$ by applying the quotient rule formula $\frac{dy}{dx} = \frac{v\frac{du}{dx} - u\frac{dv}{dx}}{v^2}$

Be careful using the formula – because of the **minus sign** in the **numerator**, the **order** of the functions is important

Simplify the answer if straightforward or if the question requires a particular form

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Your notes

e.g. $y = \frac{x^2+1}{x-3}$ ← $u = x^2+1, \quad v = x-3$

v, $\frac{du}{dx}$, u, $\frac{dv}{dx}$

$$\frac{dy}{dx} = \frac{(x-3)(2x)-(x^2+1)(1)}{(x-3)^2}$$

$(x-3)^2$ ← v^2

$$\frac{dy}{dx} = \frac{x^2-6x-1}{(x-3)^2}$$

$f(x) = \frac{\sin x}{\cos x}$ ← $g(x) = \sin x, \quad h(x) = \cos x$

h(x), g′(x), g(x), h′(x)

$$f'(x) = \frac{(\cos x)(\cos x)-(\sin x)(-\sin x)}{(\cos x)^2}$$

$(\cos x)^2$ ← $h((x))^2$

$$= \frac{\cos^2 x + \sin^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

$f'(x) = \sec^2 x$ ← $\frac{1}{\cos x} = \sec x$

$\frac{\sin x}{\cos x} = \tan x$ SO THIS IS THE DERIVATIVE OF $\tan x$

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Examiner Tip

- The quotient rule formula is **not** on the list of formulas page – you have to memorise it
  - however if you do forget it in an exam, you could rewrite $y = \frac{u}{v}$ as $y = u \times v^{-1}$ and then use the **product rule**
  - (the quotient rule is really doing exactly this)
- Be careful using the formula – because of the minus sign in the numerator the order of the functions is important!


Your notes

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Worked example

Your notes

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Your notes

Differentiate $f(x) = (\cos 2x)\left(x^2 + 2x + 1\right)^{-1}$ with respect to x.

$f(x) = \dfrac{\cos 2x}{x^2 + 2x + 1}$ ← REWRITE AS $f(x) = \dfrac{g(x)}{h(x)}$

$g(x) = \cos 2x$ $\quad$ $h(x) = x^2 + 2x + 1$

$g'(x) = -2\sin 2x$ $\quad$ $h'(x) = 2x + 2$

CHAIN RULE – AND REMEMBER MINUS SIGN WHEN DIFFERENTIATING cos

$$f'(x) = \frac{(x^2 + 2x + 1)(-2\sin 2x) - (\cos 2x)(2x + 2)}{(x^2 + 2x + 1)^2}$$

$f'(x) = \dfrac{h(x)g'(x) - g(x)h'(x)}{(h(x))^2}$

$$f'(x) = \frac{-2\sin 2x(x^2 + 2x + 1) - 2\cos 2x(x + 1)}{(x^2 + 2x + 1)^2}$$

$$= \frac{-2\sin 2x(x + 1)^2 - 2\cos 2x(x + 1)}{(x + 1)^4}$$ ← $x^2 + 2x + 1 = (x + 1)^2$

$$f'(x) = \frac{-2(\cos 2x + (x + 1)\sin 2x)}{(x + 1)^3}$$ ← SIMPLIFY IT

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Your notes

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Applications of Differentiation

Your notes

Finding Gradients

How do I use the derivative to find the gradient of a curve?

- The **gradient of a curve** at a point is the **gradient of the tangent** to the curve at that point
- To find the gradient of a curve, at any point on the curve
  - **differentiate** to find $\frac{dy}{dx}$ (unless $\frac{dy}{dx}$ is already known)
  - substitute the ***x*-coordinate** of the point into the derivative $\frac{dy}{dx}$ and evaluate

e.g. FIND THE GRADIENT OF THE CURVE $y = x - \frac{1}{x}$ AT THE POINT $(2, \frac{3}{2})$

$y = x - x^{-1}$

FIND DERIVATIVE

$\frac{dy}{dx} = 1-(-1)x^{-1-1} = 1+x^{-2} = 1+\frac{1}{x^2}$

WHEN $x = 2$,

PUT x−COORDINATE INTO DERIVATIVE

$\frac{dy}{dx} = 1+\frac{1}{2^2} = \frac{5}{4}$

THE GRADIENT AT $(2, \frac{3}{2})$ IS $\frac{5}{4}$

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How do I find the approximate change in *y* as *x* increases?

- $\text{gradient} = \frac{\text{change in } y}{\text{change in } x}$ so, for **small** changes you can write
  - $\text{change in } y = \text{gradient} \times \text{change in } x$
- For example, if the gradient of $y = x - \frac{1}{x}$ at $x = 2$ is $\frac{5}{4}$
  - what is the approximate change in y as x increases from 2 to $2 + h$, where h is small?

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- change in y = gradient $\times$ change in $x = \frac{5}{4}h$

Your notes

Examiner Tip

- Read the question carefully; sometimes you are given $\frac{dy}{dx}$ and so don't need to differentiate initially - don't just automatically differentiate the first thing you see!
- The following mean the same thing:
  - "Find the gradient of the curve at $x = 2$"
  - "Find the gradient of the tangent at $x = 2$"
    - the tangent gradient = curve gradient at that point
  - "Find the **rate of change** of y with respect to x at $x = 2$"

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Your notes

Worked example

A curve has the equation $y = \frac{4}{3}x^3 + 3x - 8$.

(a) Find the gradient of the curve when $x = 2$.

y is already in a form that can be differentiated

$$\frac{dy}{dx} = 4x^2 + 3$$

Substitute $x = 2$ into $\frac{dy}{dx}$

$$\frac{dy}{dx} = 4 \times 2^2 + 3 = 19$$

The gradient of the curve at $x = 2$ is 19

(b) Work out the possible values of x for which the rate of change of y with respect to x is 4.

"Rate of change" is another way of describing the derivative

$$\frac{dy}{dx} = 4$$

$$4x^2 + 3 = 4$$

Solve this equation to find x
Note that it is quadratic equation so it could have up to two solutions
The question refers to 'values' implying there is (or could be) more than one value for x

$$4x^2 = 1$$

$$x^2 = \frac{1}{4}$$

$$x = \pm\sqrt{\frac{1}{4}}$$

$$x = \pm\frac{1}{2}$$

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The possible values of x, that give a rate of change of 4, are $x = \frac{1}{2}$ and $x = -\frac{1}{2}$


Your notes

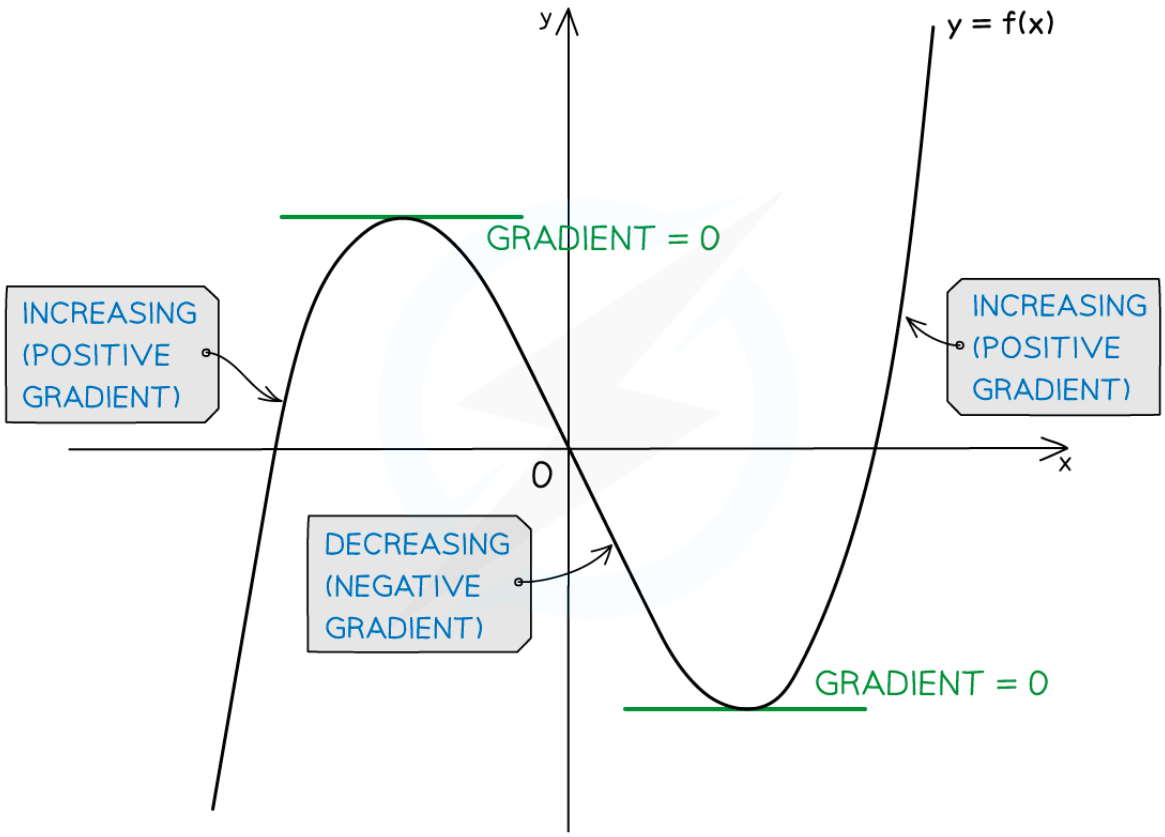
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Increasing & Decreasing Functions

Your notes

What are increasing and decreasing functions?

- A function is **increasing** when $\frac{dy}{dx} > 0$ **(the gradient is positive)**
  - This means graph of a function goes **up** as x increases
- A function is **decreasing** when $\frac{dy}{dx} < 0$ **(the gradient is negative)**
  - This means graph of a function goes **down** as x increases



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How do I find where functions are increasing or decreasing?

- To identify the **intervals** on which a function is increasing or decreasing

STEP 1

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Find the derivative **f'(x)**

STEP 2
Solve the inequalities
$f'(x) > 0$ (for increasing intervals) and/or
$f'(x) < 0$ (for decreasing intervals)

- Most functions are a combination of **increasing, decreasing** and **stationary**
  - a range of values of x (**interval**) is given where a function satisfies each condition
  - e.g. The function $f(x) = x^2$ has **derivative** $f'(x) = 2x$ so
    - $f(x)$ is **decreasing** for $x < 0$
    - $f(x)$ is **stationary** at $x = 0$
    - $f(x)$ is **increasing** for $x > 0$
- To identify the intervals (the range of x values) for which a curve is increasing or decreasing you need to:
  1. Find the derivative $\frac{dy}{dx}$
  2. Solve the inequalities $\frac{dy}{dx} > 0$ (for increasing intervals) or $\frac{dy}{dx} < 0$ (for decreasing intervals)

Examiner Tip

- In an exam, if you need to show a function is increasing or decreasing you can use either strict (<, >) or non-strict (≤, ≥) inequalities
  - You will get the marks either way in this course

Your notes

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Your notes

Worked example

For what values of x is $y = 2x^3 - 3x^2 + 5$ a decreasing function?

The function is decreasing when its gradient $\left(\frac{dy}{dx}\right)$ is less than 0.

Find the derivative of the function by differentiating.

$$\frac{dy}{dx} = 6x^2 - 6x$$

Solve the inequality $\frac{dy}{dx} < 0$ to find the set of values where the gradient is negative.

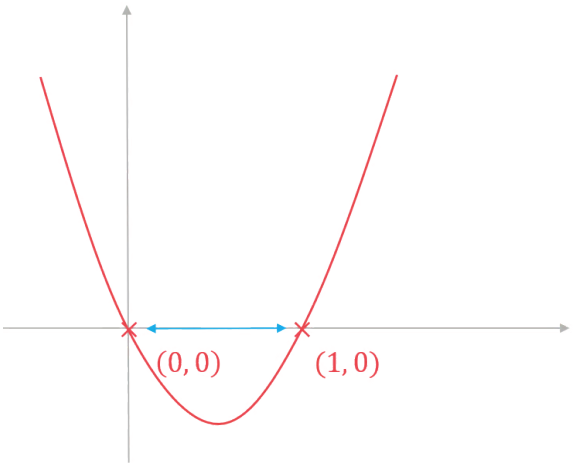
$$6x^2 - 6x < 0$$

Factorise.

$$6x(x - 1) < 0$$

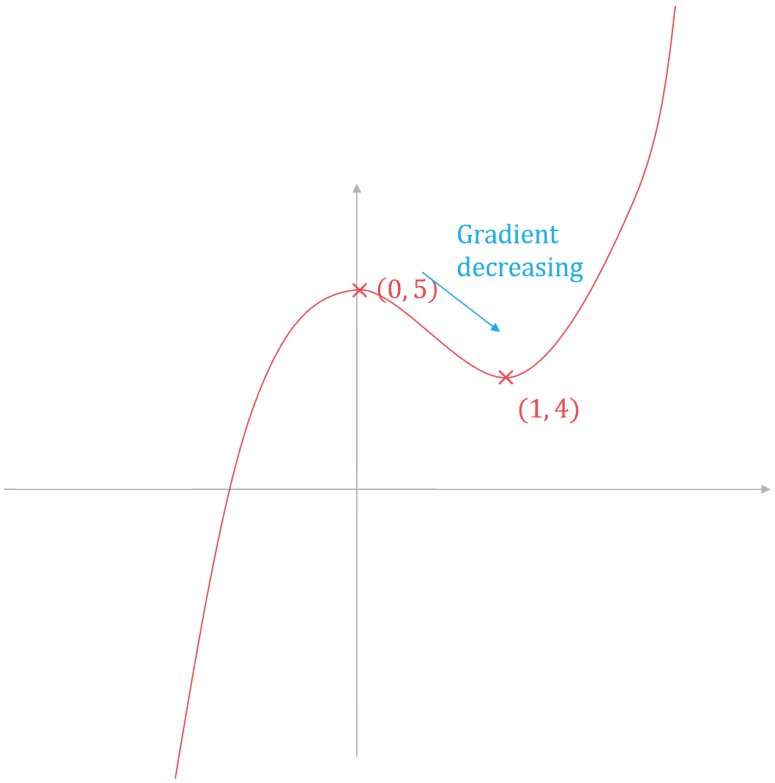
The solutions to $\frac{dy}{dx} = 0$ are $x = 0$ and $x = 1$. Find the correct way around for the inequalities by considering the graph of $\frac{dy}{dx}$. The graph is a positive quadratic, so the function is negative between the values of 0 and 1 (where it is below the x-axis).

Considering a sketch of the graph of the gradient function may help you see this.



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You can check your answers by considering a sketch of the original function, it should be decreasing at the point where $0 < x < 1$.



$0 < x < 1$

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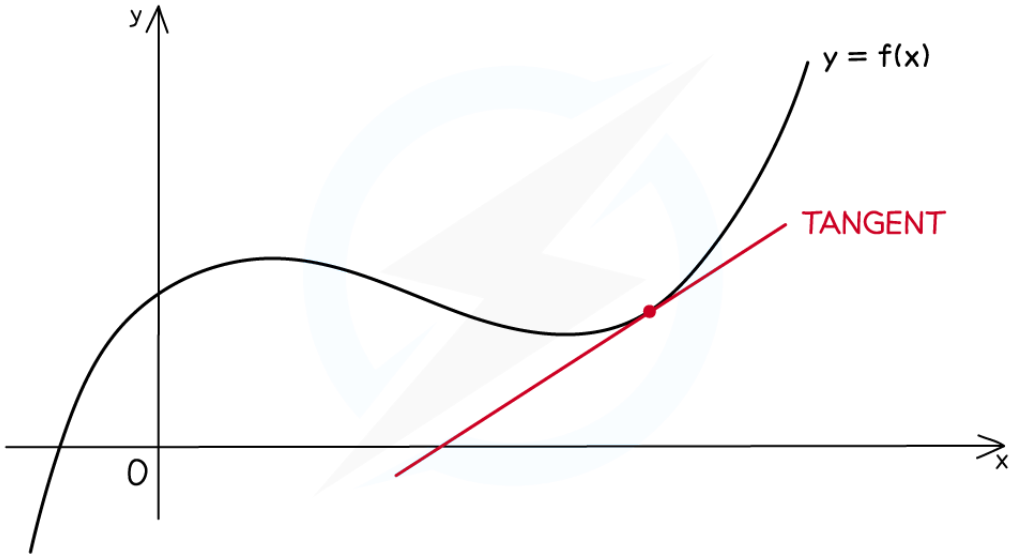
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Tangents & Normals

Your notes

What is a tangent?

- At any point on the graph of a (non-linear) **function**, the **tangent** is the straight line that **touches** the graph at a point **without crossing** through it
- Its **gradient** is given by the **derivative function**



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How do I find the equation of a tangent?

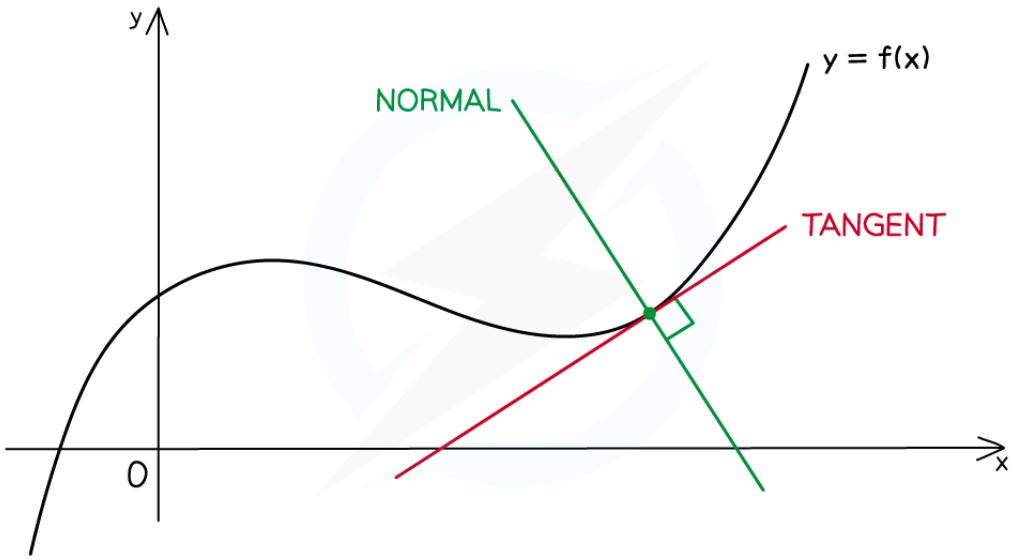
- To find the **equation of a straight line**, a **point** and the **gradient** are needed
- The **gradient**, m, of the **tangent** to the function $y = \mathrm{f}(x)$ at (x_1, y_1) is $\mathbf{f'(x_1)}$
  - You can find this by differentiating the function, and then substituting the x-coordinate of the point into the derivative
- Therefore find the **equation** of the **tangent** to the function $y = \mathrm{f}(x)$ at the point (x_1, y_1) by substituting the gradient, $\mathrm{f}'(x_1)$, and point (x_1, y_1) into $y - y_1 = m(x - x_1)$, giving:
  - $y - y_1 = \mathrm{f}'(x_1)(x - x_1)$
- (You could also substitute into $y = mx + c$)

What is a normal?

- At any point on the graph of a (non-linear) function, the **normal** is the straight line that passes through that point and is **perpendicular** to the **tangent**

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Your notes



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How do I find the equation of a normal?

- The gradients of two perpendicular lines are **negative reciprocals**
  - This means that if m_1 is the gradient of the first line and m_2 is the gradient of a line perpendicular to the first line, then $m_2 = \frac{-1}{m_1}$
  - Rearranging the formula above, $m_1 \times m_2 = -1$ is a useful way to test whether two lines are perpendicular
- Therefore **gradient** of the **normal** to the function $y = f(x)$ at (x_1, y_1) is $\frac{-1}{f'(x_1)}$
- Find the **equation** of the **normal** to the function $y = f(x)$ at the point (x_1, y_1) by using

$$y - y_1 = \frac{-1}{f'(x_1)}(x - x_1)$$

  - (or $y = \frac{-1}{f'(x_1)}x + c$)

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Examiner Tip

- To be successful in this topic, first make sure you are confident with finding the equation of a straight line!



Your notes

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Your notes

Worked example

The function $f(x)$ is defined by

$$f(x) = 2x^4 + \frac{3}{x^2} \qquad x \neq 0$$

a) Find an equation for the tangent to the curve $y = f(x)$ at the point where $x = 1$, giving your answer in the form $y = mx + c$.

First find the derivative $f'(x)$ by differentiating
Start by rewriting $f(x)$ as powers of x

$$f(x) = 2x^4 + 3x^{-2}$$

Now differentiate

$$f'(x) = 8x^3 - 6x^{-3}$$

Now substitute $x = 1$ into $f'(x)$ to find the gradient of the tangent

$$\begin{aligned} f'(1) &= 8(1)^3 - 6(1)^{-3} \\ &= 8 - 6 \\ &= 2 \end{aligned}$$

We also need the y-coordinate, so substitute $x = 1$ into $f(x)$ also

$$\begin{aligned} f(1) &= 2(1)^4 + 3(1)^{-2} \\ &= 2 + 3 \\ &= 5 \end{aligned}$$

Now we can substitute the point (1, 5) and the gradient, 2, into $y - y_1 = m(x_1)(x - x_1)$

$$y - 5 = 2(x - 1)$$

Note that we are asked for the final answer in the form $y = mx + c$, so rearrange to this form

$$y - 5 = 2x - 2$$

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$y = 2x + 3$

Your notes

b) Find an equation for the normal at the point where $x = 1$, giving your answer in the form $ax + by + d = 0$, where a, b and d are integers.

We already have the gradient of the tangent; the gradient of the normal is $\frac{-1}{m}$ where m is the gradient of the tangent

$$\text{gradient of normal} = \frac{-1}{2}$$

Substitute the point (1, 5) from part (a) and the gradient of the normal into $y - y_1 = m(x - x_1)$

$$y - 5 = \left(\frac{-1}{2}\right)(x - 1)$$

And rearrange into the form required

$$y - 5 = -\frac{1}{2}x + \frac{1}{2}$$

$$\frac{1}{2}x + y - 5\frac{1}{2} = 0$$

Note that a, b and d must be integers so multiply by 2 to clear the fractions

$$x + 2y - 11 = 0$$

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Second Order Derivatives

Your notes

Second Order Derivatives

What is the second order derivative of a function?

- If you **differentiate the derivative** of a function (i.e. differentiate the function a second time) you get the **second order derivative** of the function
  - The second order derivative can be referred to simply as the second derivative
- We can write the second derivative as $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
- Note the position of the powers of 2
  - **d**ifferentiating twice (so $\mathbf{d^2}$) with respect to x twice (so x^2)
- A **first derivative** is the **rate** of **change** of a function (the gradient)
  - a **second order derivative** is the **rate of change of the rate of change** of a function
    - i.e. the rate of change of the function's gradient
  - A positive second derivative means the gradient is increasing
    - For instance in a u-shape, the gradient is changing from negative to positive
  - A negative second derivative means the gradient is decreasing
    - For instance in an n-shape, the gradient is changing from positive to negative
- Second order derivatives can be used to test whether a point is a minimum or maximum
- To find a second derivative, you simply **differentiate twice**!
  - It is important to write down your working with the correct notation, so you know what each expression means
  - For example
    - $y = 5x^3 + 10x^2$
    - $\frac{\mathrm{d}y}{\mathrm{d}x} = 15x^2 + 20x$
    - $\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = 30x + 20$

Examiner Tip

- Even if you think you can find the second derivative in your head and write it down, make sure you write down the first derivative as well
  - If you make a mistake, you will most likely get marks for finding the first derivative

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Your notes

Worked example

Work out $\frac{d^2y}{dx^2}$ when

(a) $y = x^5 - 2x^3 + 7x^2 + 9x - 18$

Find the first derivative of the function first by considering each term in turn.

$$\frac{dy}{dx} = 5x^4 - 6x^2 + 14x + 9$$

Find the second derivative, $\frac{d^2y}{dx^2}$ by differentiating each term in the first derivative.

$$\frac{d^2y}{dx^2} = 20x^3 - 12x + 14$$

(b) $y = \frac{3x + 7}{x^4}$

To find the first derivative of the function, begin by separating the terms in the fraction.

$$y = \frac{3x}{x^4} + \frac{7}{x^4}$$

Rewrite each term using index notation so that they are in a form that can be differentiated.

$$y = 3x(x^{-4}) + 7x^{-4} = 3x^{-3} + 7x^{-4}$$

Find the first derivative of the function first by differentiating each term in turn.

$$\frac{dy}{dx} = -9x^{-4} - 28x^{-5}$$

Find the second derivative, $\frac{d^2y}{dx^2}$ by differentiating each term in the first derivative.

$$\frac{d^2y}{dx^2} = -9(-4)x^{-5} - 28(-5)x^{-6} = 36x^{-5} + 140x^{-6}$$

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You can turn the second derivative back into the same format as the original function by rewriting as a fraction.

$$\frac{d^2y}{dx^2} = \frac{36}{x^5} + \frac{140}{x^6} = \frac{36x + 140}{x^6}$$

$$\frac{d^2y}{dx^2} = \frac{36x + 140}{x^6}$$

Your notes

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Stationary Points & Turning Points

What are stationary points?


Your notes

- A **stationary point** is any point on a curve where the **gradient** is **zero**
- To find stationary points of a curve

Step 1

Find the first derivative $\frac{\mathrm{d}y}{\mathrm{d}x}$

Step 2

Solve $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ to find the x-coordinates of any stationary points

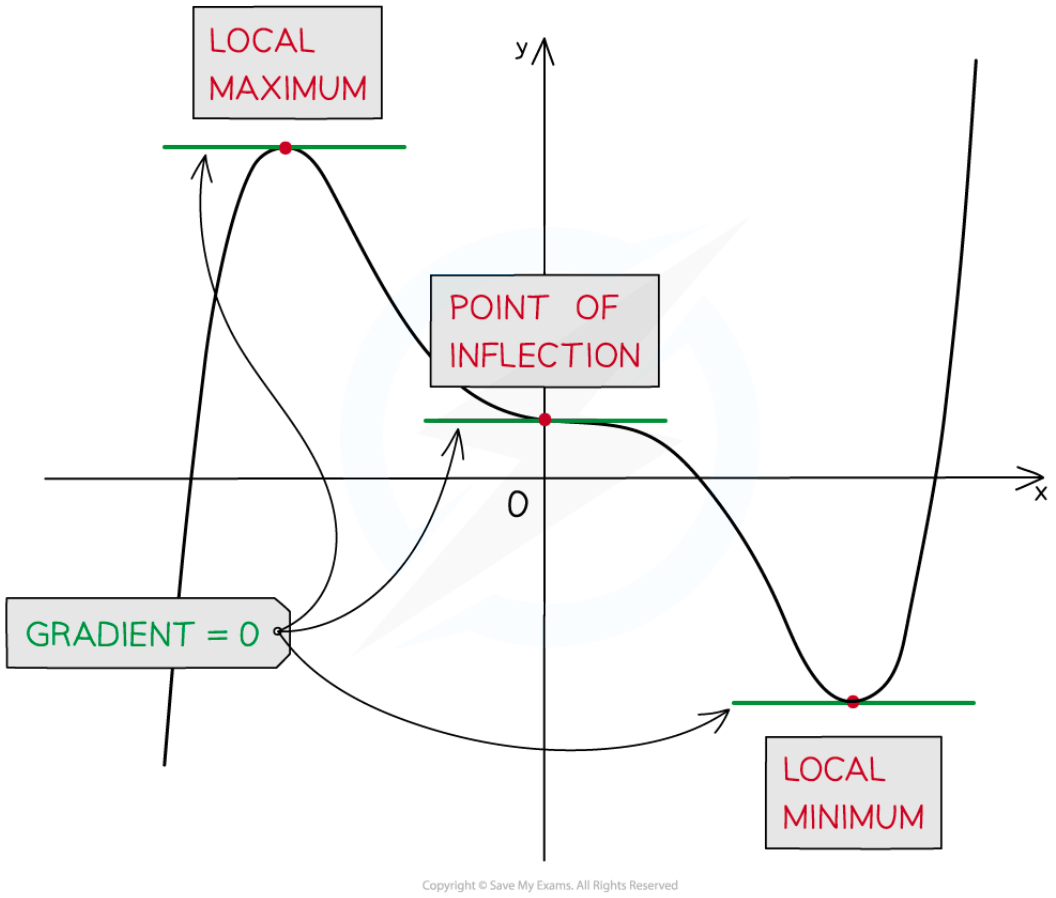
Step 3

Substitute those x-coordinates into the equation of the curve to find the corresponding y-coordinates

- A stationary point may be either a **local minimum**, a **local maximum**, or a **point of inflection**

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Your notes



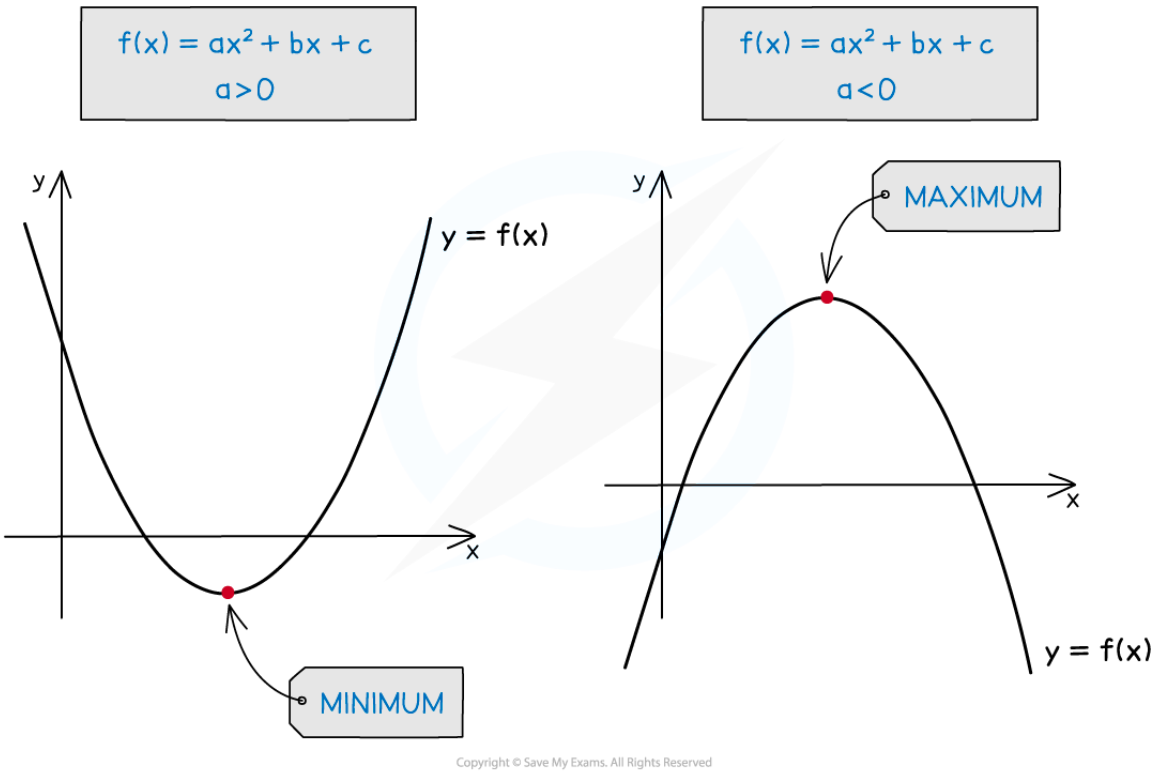
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Stationary points on quadratics

- The graph of a **quadratic** function only has a **single stationary point**
- For a **positive** quadratic this is the **minimum**; for a **negative** quadratic it is the **maximum**
  - No need to talk about 'local' here, as it is the overall minimum/maximum for the whole curve

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Your notes



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- The ***y*** value/coordinate of the stationary point is therefore the minimum or maximum value of the quadratic function
- For quadratics especially, minimum and maximum points are often referred to as **turning points**

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Worked example

Find the stationary point of

$$y = x^2 - 2x$$

and explain why it will be a minimum point.

Find the derivative.

$$\frac{dy}{dx} = 2x - 2$$

Solve $\frac{dy}{dx} = 0$

$$2x - 2 = 0$$
$$2x = 2$$
$$x = 1$$

Find the corresponding y-coordinate.

$$x = 1,\ y = (1)^2 - 2(1) = -1$$

Write the answer as a coordinate.

The stationary point is (1, -1)

To explain why it is a minimum point, consider the shape of the quadratic.

(1, -1) is a minimum point because it is a positive quadratic.

Your notes

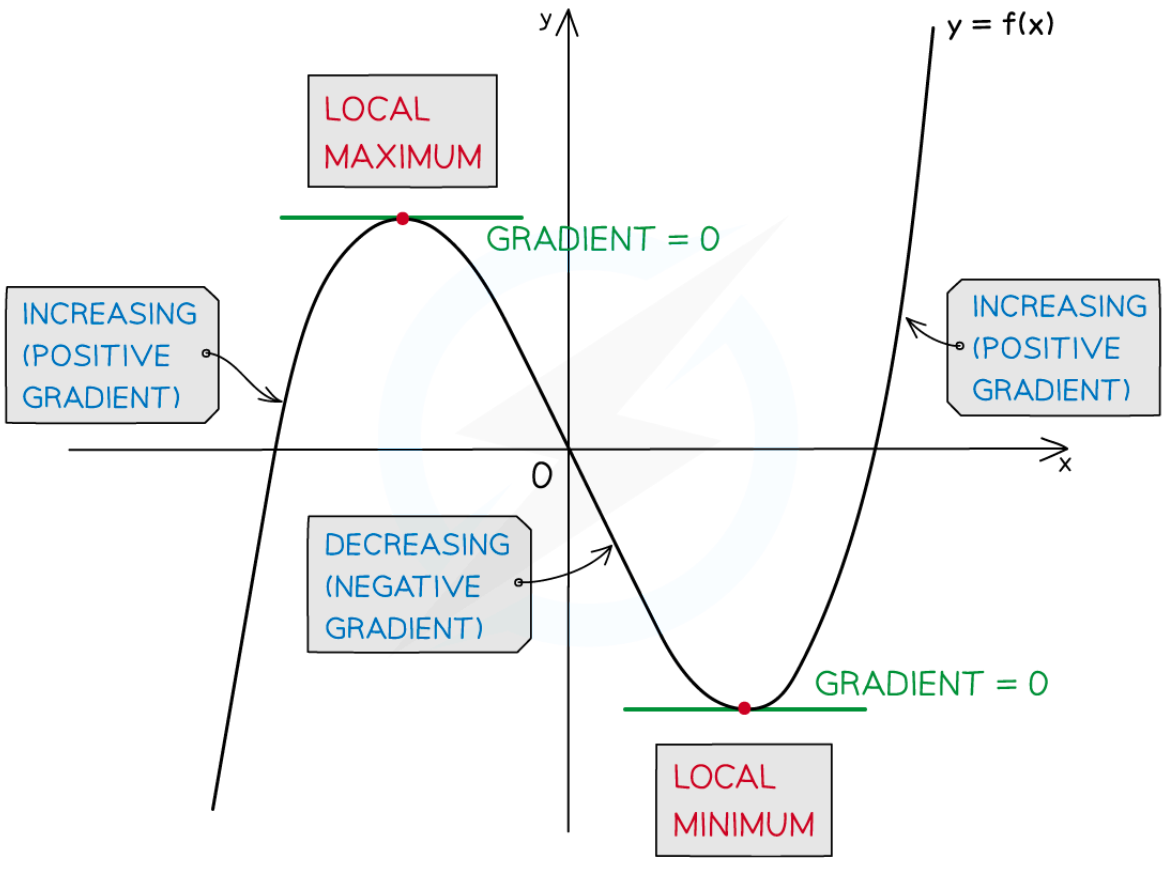
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Testing for Local Minimum & Maximum Points

Your notes

How do I determine the nature of stationary points on a curve?

- For a graph there are two ways to determine the nature of its stationary points
- **Method A**
  - Compare the signs of the **first derivative,** $\frac{dy}{dx}$ **,** (positive or negative) a little bit **to either side of the stationary point**
    - e.g. if the stationary point is at x=2 then you could find the gradient at x=1.9 and x=2.1
  - Compare the **signs** (positive or negative) of the derivatives on the left and right of the stationary point
    - If the derivatives are **negative** on the **left** and **positive** on the **right**, the point is a **local minimum (a u-shape)**
    - If the derivatives are **positive** on the **left** and **negative** on the **right**, the point is a **local maximum (an n-shape)**



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- **Method B**
  - Look at the sign of the **second derivative** (positive or negative) **at the stationary point**
  - Find the second derivative $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$
  - For each stationary point find the value of $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ at the stationary point
    - i.e. substitute the x-coordinate of the stationary point into $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ and evaluate
  - If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is **positive** then the point is a **local minimum**
  - If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is **negative** then the point is a **local maximum**
  - If $\frac{\mathrm{d}^2y}{\mathrm{d}x^2}$ is **zero** then the point could be a **local minimum**, a **local maximum** OR a **point of inflection**
    - In this case you will need to use method A instead


Your notes

Examiner Tip

- Usually, using the second derivative (Method B above) is a much quicker way of determining the nature of a stationary point
- Sketching the curve can also help

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Your notes

Worked example

Find the stationary points of

$$y = x^2(2x^2 - 4)$$

and determine the nature of each.

Start by expanding the brackets in the expression for y

$$y = 2x^4 - 4x^2$$

Step 1
Find the first derivative

$$\frac{dy}{dx} = 8x^3 - 8x$$

Step 2
Solve $\frac{dy}{dx} = 0$

$$8x^3 - 8x = 0$$

Divide through by 8

$$x^3 - x = 0$$

Fully factorise, spotting the difference of two squares

$$x(x^2 - 1) = 0$$
$$x(x - 1)(x + 1) = 0$$

Solve to find the x-coordinates of the stationary (turning) points

$$x_1 = 0, x_2 = 1, x_3 = -1$$

Step 3
Find the corresponding y-coordinates

$$x_1 = 0, \quad y_1 = (0)^2(2(0)^2 - 4) = 0$$
$$x_2 = 1, \quad y_2 = (1)^2(2(1)^2 - 4) = -2$$
$$x_3 = -1, \quad y_3 = (-1)^2(2(-1)^2 - 4) = -2$$

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Write the answers as coordinates, being careful to correctly match x's and y's

The stationary points are (0, 0), (1, -2) and (-1, -2)

To find the nature of each stationary point, use Method B

Step 4

Find the second derivative

$$\frac{d^2y}{dx^2} = 24x^2 - 8$$

Step 5

Evaluate the second derivative at each stationary point

$$x = 0, \frac{d^2y}{dx^2} = 24(0)^2 - 8 = -8 \ (<0)$$

$$x = 1, \frac{d^2y}{dx^2} = 24(1)^2 - 8 = 16 \ (>0)$$

$$x = -1, \frac{d^2y}{dx^2} = 24(-1)^2 - 8 = 16 \ (>0)$$

The nature of each stationary point is now determined

(0, 0) is a local minimum point
(1, -2) is a local minimum point
(-1, -2) is a local minimum point

Your notes

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Modelling with Differentiation

Your notes

Modelling with Differentiation

How is differentiation used in modelling questions?

- **Derivatives** can be calculated for any variables – not just ***y*** and ***x***
- The derivative is a formula giving the **rate of change** of one variable with **respect** to the other variable
  - For example if $A = 4\pi r^2$ then $\frac{\mathrm{d}A}{\mathrm{d}r} = 8\pi r$
  - $\frac{\mathrm{d}A}{\mathrm{d}r}$ is the **rate of change** of A with **respect** to r
- The phrase 'increasing at a rate of' means the **rate of change** of one variable with **respect** to time
  - $\frac{\mathrm{d}}{\mathrm{d}t}$
- **Differentiation** can be used to find **maximum** and **minimum** points of a function
  - In modelling, this is called **optimisation**
  - Second derivative tests help to determine is the point is a maximum or minimum

Examiner Tip

- Read the question carefully to determine which variables you will need to use
  - The question may give you a formula to help you

Worked example

The volume, V, of a sphere of radius r is given by $V = \frac{4}{3}\pi r^3$.

Find the rate of change of the volume with respect to the radius.

Differentiate the formula given for the volume of a sphere.

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 3 \times \frac{4}{3}\pi \times r^2$$

$$\frac{\mathrm{d}V}{\mathrm{d}r} = 4\pi r^2$$

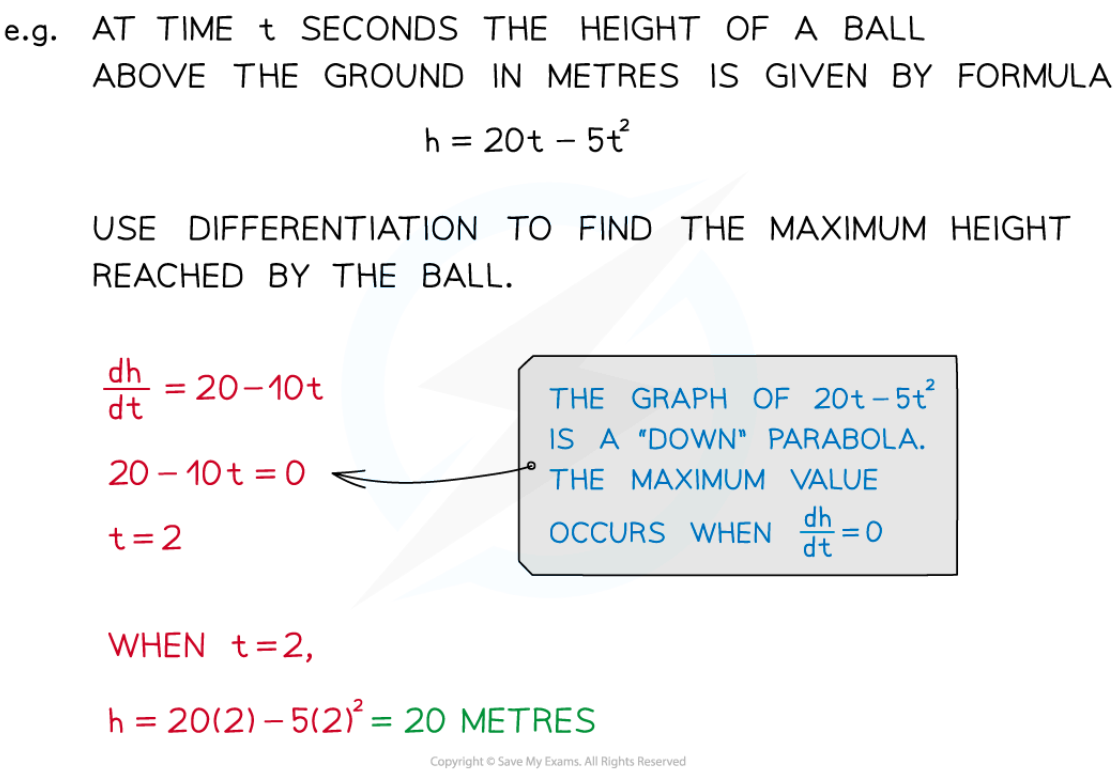
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Optimisation

What is optimisation?

Your notes

- In general, optimisation is finding the best way to do something
- In mathematics, optimisation is finding the maximum or minimum output of a function
  - For example, finding the maximum possible profit or minimum costs
- Differentiation can be used to **solve optimisation problems** in **modelling** questions
  - For example you may want to
    - **Maximise** the **volume** of a **container**
    - **Minimise** the **amount** of **fuel** used



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Examiner Tip

- Exam questions on this topic will often be divided into two parts:
  - First a 'Show that...' part where you derive a given formula from the information in the question
  - And then a 'Find...' part where you use differentiation to answer a question about the formula
- Even if you can't answer the first part you can still use the formula to answer the second part

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Your notes

Worked example

A cuboid has length $4x$ cm, width x cm, and height $\left(\frac{3}{x}-5\right)$ cm.

(a) Show that the volume, V cm^3 is given by $V=12x-20x^2$.

The volume of a cuboid is "$V = length \times width \times height$"

$$\therefore V=4x\times x\times\left(\frac{3}{x}-5\right)$$

Expand and simplify

$$V=4x^2\left(\frac{3}{x}-5\right)$$

$$V=\frac{12x^2}{x}-20x^2$$

$$\therefore V=12x-20x^2$$

(b) Find the maximum volume of the cuboid.

Differentiate V with respect to x

$$\frac{dV}{dx}=12-40x$$

At the maximum volume, $\frac{dV}{dx}=0$

$$\therefore 12-40x=0$$

Solve for x

$$40x=12$$

$$x=\frac{12}{40}=0.3$$

So the value of x, at the maximum volume is 0.3
Find the maximum volume by substituting $x = 0.3$ in to the formula for V

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$$V = 12(0.3) - 20(0.3)^2 = 1.8$$

The maximum volume of the cuboid is 1.8 cm^3

(c) Prove that your answer is a maximum value.

Using the second derivative is usually the easiest way to find the nature of a stationary point

$$\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = -40 \ (<0)$$

The value of the second derivative (at $x = 0.3$) is negative

Therefore V = 1.8 cm^3 is a maximum volume

Your notes

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Connected Rates of Change


Your notes

Connected Rates of Change

What are connected rates of change?

- In situations involving more than two variables you can use the **chain rule** to connect multiple rates of change into a single equation
- Look out for the word "**rate**"
  - This implies a **derivative** is involved
  - e.g. The height is **increasing** at a **rate of 4 cm per second** means $\frac{\mathrm{d}h}{\mathrm{d}t} = 4$
- If the variable is **decreasing** then the rate will be **negative**
  - e.g. The height is **decreasing** at a rate of 4 cm per second means $\frac{\mathrm{d}h}{\mathrm{d}t} = -4$

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Your notes

e.g. THE VOLUME V ,OF, A CUBE IS INCREASING AT A CONSTANT RATE OF 3 cm³ PER SECOND. ASSUMING THAT THE SHAPE REMAINS CUBICAL AT ALL TIMES, FIND THE RATE OF CHANGE OF THE SIDE LENGTH, S, IN cm PER SECOND AT THE MOMENT WHEN S = 5 cm.

$\frac{dV}{dt} = 3\ cm^3/sec$

WE KNOW THIS

WE WANT TO KNOW THIS

BY THE CHAIN RULE $\frac{dV}{dt} = \frac{dV}{ds} \times \frac{ds}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

FOR A CUBE, $V = s^3$

SO $\frac{dV}{ds} = 3s^2$

USE DIFFERENTIATION TO GET THE 'MISSING' DERIVATIVE FROM THE CHAIN RULE EQUATION

SO $\frac{ds}{dt} = \frac{\frac{dV}{dt}}{\frac{dV}{ds}} = \frac{3}{3s^2} = \frac{1}{s^2}$

REARRANGE

WHEN s = 5,

$\frac{ds}{dt} = \frac{1}{5^2} = \frac{1}{25}\ cm/sec$

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Examiner Tip

- These problems can involve a lot of letters
  - be sure to keep track of what they all refer to
- Be especially sure that you are clear about which letters are **variables** and which are **constants**
  - these behave very differently when differentiation is involved!

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Worked example

Your notes

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Your notes

In a manufacturing process, plastic spheres are produced in such a way that the volume, V, of a sphere increases at a constant rate of $10\ \text{cm}^3$ per second. Find the rate of change of the surface area, A, of a sphere at the moment when the surface area is equal to $32\pi\ \text{cm}^2$.

$\frac{dV}{dt} = 10\ \text{cm}^3/\text{sec}$

WE KNOW THIS

WE WANT TO KNOW THIS

BY THE CHAIN RULE $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

USE THE CHAIN RULE TO CONNECT WHAT YOU KNOW AND WHAT YOU WANT TO KNOW

$V = \frac{4}{3}\pi r^3$ (WHERE r IS THE RADIUS)

WRITE V IN TERMS OF A

$A = 4\pi r^2$

$A^{\frac{3}{2}} = (4\pi r^2)^{\frac{3}{2}} = 8\pi^{\frac{3}{2}} r^3 = 8\pi\sqrt{\pi}\, r^3$

$\frac{8\pi\sqrt{\pi}\, r^3}{6\sqrt{\pi}} = \frac{4}{3}\pi r^3$ SO $V = \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}}$

AND $\frac{dV}{dA} = \frac{3}{2} \times \frac{1}{6\sqrt{\pi}} A^{\frac{3}{2}-1} = \frac{1}{4\sqrt{\pi}} A^{\frac{1}{2}}$

DIFFERENTIATE TO FIND THE 'MISSING' DERIVATIVE

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Your notes

SO $\frac{dV}{dt} = \frac{dV}{dA} \times \frac{dA}{dt}$

SUBSTITUTE AND REARRANGE

$10 = \frac{1}{4\sqrt{\pi}} A^{\frac{1}{2}} \left(\frac{dA}{dt} \right)$

AND $\frac{dA}{dt} = \frac{40\sqrt{\pi}}{A^{\frac{1}{2}}} = 40\sqrt{\frac{\pi}{A}}$

WHEN $A = 32\pi$,

$\frac{dA}{dt} = 40\sqrt{\frac{\pi}{32\pi}} = \frac{40}{\sqrt{32}} = 5\sqrt{2}\ cm^2/sec$

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