CIE IGCSE Maths: Extended

Your notes

Quadratic Equations

Contents

- Solving Quadratics by Factorising
- Quadratic Formula
- Completing the Square
- Quadratic Equation Methods

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Solving Quadratics by Factorising

Your notes

Solving Quadratics by Factorising

How do I solve a quadratic equation using factorisation?

- Rearrange it into the form $ax^2 + bx + c = 0$
  - zero must be on one side
    - it is easier to use the side where a is positive
- Factorise the quadratic and solve each bracket equal to zero
  - If $(x + 4)(x - 1) = 0$, then either $x + 4 = 0$ or $x - 1 = 0$
    - Because if $A \times B = 0$, then either $A = 0$ or $B = 0$
- To solve $(x - 3)(x + 7) = 0$
  - ...solve "**first bracket = 0**":
    - $x - 3 = 0$
    - add 3 to both sides: $x = 3$
  - ...and solve "**second bracket = 0**"
    - $x + 7 = 0$
    - subtract 7 from both sides: $x = -7$
  - The **two solutions** are $x = 3$ or $x = -7$
    - The solutions have the opposite signs to the numbers in the brackets
- To solve $(2x - 3)(3x + 5) = 0$
  - ...solve "first bracket = 0"
    - $2x - 3 = 0$
    - add 3 to both sides: $2x = 3$
    - divide both sides by 2: $x = \frac{3}{2}$
  - ...solve "second bracket = 0"
    - $3x + 5 = 0$
    - subtract 5 from both sides: $3x = -5$
    - divide both sides by 3: $x = -\frac{5}{3}$
  - The two solutions are $x = \frac{3}{2}$ or $x = -\frac{5}{3}$
- To solve $x(x - 4) = 0$
  - it may help to think of x as $(x - 0)$ or (x)
  - ...solve "first bracket = 0"
    - $(x) = 0$, so $x = 0$
  - ...solve "second bracket = 0"

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- $x - 4 = 0$
- add 4 to both sides: $x = 4$

- The **two solutions** are $x = 0$ or $x = 4$
  - It is a common **mistake** to divide both sides by x at the beginning - you will **lose a solution** (the $x = 0$ solution)

Your notes

Exam Tip

- Use a calculator to check your final solutions!
  - Calculators also help you to factorise (if you're struggling with that step)
  - A calculator gives solutions to $6x^2 + x - 2 = 0$ as $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
    - "Reverse" the method above to factorise!
    - $6x^2 + x - 2 = (3x + 2)(2x - 1)$
  - Warning: a calculator gives solutions to $12x^2 + 2x - 4 = 0$ as $x = -\frac{2}{3}$ and $x = \frac{1}{2}$
    - But $12x^2 + 2x - 4 \neq (3x + 2)(2x - 1)$ as these brackets expand to $6x^2 + \ldots$ not $12x^2 + \ldots$
    - Multiply by 2 to correct this
    - $12x^2 + 2x - 4 = 2(3x + 2)(2x - 1)$

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

(a) Solve $(x-2)(x+5)=0$

Set the first bracket equal to zero

$x - 2 = 0$

Add 2 to both sides

$x = 2$

Set the second bracket equal to zero

$x + 5 = 0$

Subtract 5 from both sides

$x = -5$

Write both solutions together using "or"

$x = 2$ or $x = -5$

(b) Solve $(8x+7)(2x-3)=0$

Set the first bracket equal to zero

$8x + 7 = 0$

Subtract 7 from both sides

$8x = -7$

Divide both sides by 8

$$x = -\frac{7}{8}$$

Set the second bracket equal to zero

$2x - 3 = 0$

Add 3 to both sides

Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

$2x = 3$

Divide both sides by 2

$$x = \frac{3}{2}$$

Write both solutions together using "or"

$$x = -\frac{7}{8} \text{ or } x = \frac{3}{2}$$

(c) Solve $x(5x - 1) = 0$

Do not divide both sides by x (this will lose a solution at the end)
Set the first "bracket" equal to zero

$(x) = 0$

Solve this equation to find x

$x = 0$

Set the second bracket equal to zero

$5x - 1 = 0$

Add 1 to both sides

$5x = 1$

Divide both sides by 5

$$x = \frac{1}{5}$$

Write both solutions together using "or"

$$x = 0 \text{ or } x = \frac{1}{5}$$

Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Quadratic Formula

Your notes

Quadratic Formula

How do I use the quadratic formula to solve a quadratic equation?

- A quadratic equation has the form:
  $ax^2 + bx + c = 0$ (as long as $a \neq 0$)
  - you need "= 0" on one side
- The **quadratic formula** is a formula that gives both solutions:
  - $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$
- Read off the values of a, b and c from the equation
- **Substitute** these into the formula
  - write this line of working in the exam
  - Put **brackets** around any **negative numbers** being substituted in
- To solve $2x^2 - 7x - 3 = 0$ using the quadratic formula:
  - $a = 2$, $b = -7$ and $c = -3$
  - $x = \dfrac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 2 \times (-3)}}{2 \times 2}$
  - Type this into a calculator or simplify by hand
    - once with + for ± and once with - for ±
  - The **solutions** are $x = 3.886$ and $x = -0.386$ (to 3 dp)
    - **Rounding** is often asked for in the question
    - The calculator also gives these solutions in **exact form** (surd form), if required
    - $x = \dfrac{7 + \sqrt{73}}{4}$ and $x = \dfrac{7 - \sqrt{73}}{4}$
  - On the non-calculator paper you will be asked to give your answers as simplified surds so make sure that the number under the square root has **no square factors**
    - If it does then **simplify the surd**!

What is the discriminant?

- The part of the formula under the square root ($b^2 - 4ac$) is called the **discriminant**
- The sign of this value tells you if there are 0, 1 or 2 solutions
  - If $b^2 - 4ac > 0$ (positive)
    - then there are 2 different solutions
  - If $b^2 - 4ac = 0$
    - then there is only 1 solution

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- sometimes called "two repeated solutions"
- If $b^2 - 4ac < 0$ (negative)
  - then there are no solutions
  - If your calculator gives you solutions with i terms in, these are "complex" and not what we are looking for
- Interestingly, if $b^2 - 4ac$ is a perfect square number (1, 4, 9, 16, ...) then the quadratic expression could have been factorised!

Can I use my calculator to solve quadratic equations?

- Yes to **check** your final answers, but a method must still be shown as above

Exam Tip

- Make sure the quadratic equation has "= 0" on the right-hand side, otherwise it needs rearranging first
- Always look for how the question wants you to leave your final answers
  - for example, correct to 2 decimal places

Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Worked example

Use the quadratic formula to find the solutions of the equation $3x^2 - 2x - 4 = 0$.
Give each solution as an exact value in its simplest form.

Write down the values of a, b and c

$$a = 3, b = -2, c = -4$$

Substitute these values into the quadratic formula, $x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

Put brackets around any negative numbers

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4 \times 3 \times (-4)}}{2 \times 3}$$

Simplify the expressions

$$x = \frac{2 \pm \sqrt{4 + 48}}{6} = \frac{2 \pm \sqrt{52}}{6}$$

Simplify the surd

$$x = = \frac{2 \pm \sqrt{4 \times 13}}{6} = \frac{2 \pm 2\sqrt{13}}{6}$$

Simplify the fraction

$$x = \frac{1 \pm \sqrt{13}}{3}$$

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Completing the Square

Your notes

Completing the Square

How can I rewrite the first two terms of a quadratic expression as the difference of two squares?

- Look at the quadratic expression $x^2 + bx + c$
- The first **two terms** can be written as the **difference of two squares** using the following rule

$$x^2 + bx \text{ is the same as } (x + p)^2 - p^2 \text{ where } p \text{ is } \textbf{half} \text{ of } b$$

- Check this is true by expanding the right-hand side
  - Is $x^2 + 2x$ the same as $(x + 1)^2 - 1^2$?
    - Yes: $(x + 1)(x + 1) - 1^2 = x^2 + 2x + 1 - 1 = x^2 + 2x$
- This works for negative values of b too
  - $x^2 - 20x$ can be written as $(x - 10)^2 - (-10)^2$ which is $(x - 10)^2 - 100$
  - A negative b does not change the sign at the end

How do I complete the square?

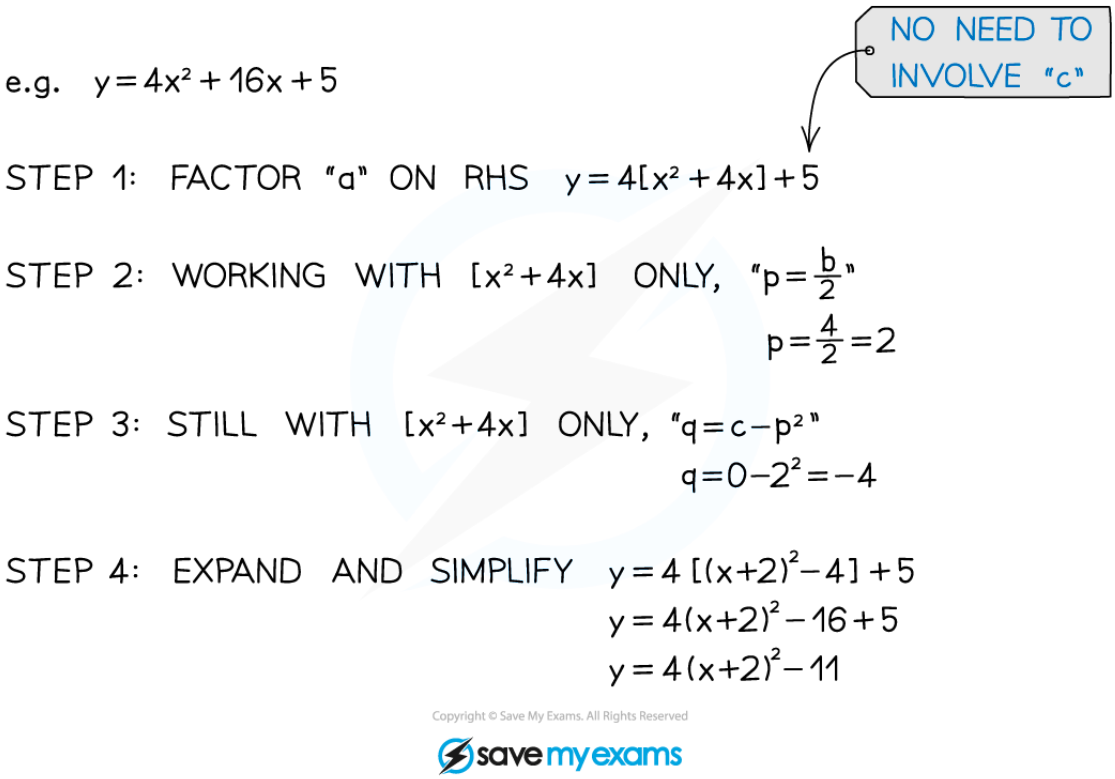
- **Completing the square** is a way to rewrite a quadratic expression in a form containing a squared-bracket
- To complete the square on $x^2 + 10x + 9$
  - Use the rule above to replace the first two terms, $x^2 + 10x$, with $(x + 5)^2 - 5^2$
  - add 9: $(x + 5)^2 - 5^2 + 9$
  - **simplify** the **numbers**: $(x + 5)^2 - 25 + 9$
  - answer: $(x + 5)^2 - 16$

How do I complete the square when there is a coefficient in front of the x^2 term?

- You first need to take a out as a factor of the x^2 and x terms only
  - $ax^2 + bx + c = a\left[x^2 + \frac{b}{a}x\right] + c$
    - Use **square-shaped brackets** here to avoid confusion with curly brackets later
- Then complete the square on the bit **inside** the square-brackets: $x^2 + \frac{b}{a}x$
  - This gives $a[(x + p)^2 - p^2] + c$
    - where p is half of $\frac{b}{a}$

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- Finally multiply this expression by the *a* outside the square-brackets and add the c
  - $a(x+p)^2 - ap^2 + c$
  - This looks far more complicated than it is in practice!
    - Usually you are asked to give your final answer in the form $a(x+p)^2 + q$
- For quadratics like $-x^2 + bx + c$, do the above with *a* = -1



Copyright © Save My Exams. All Rights Reserved

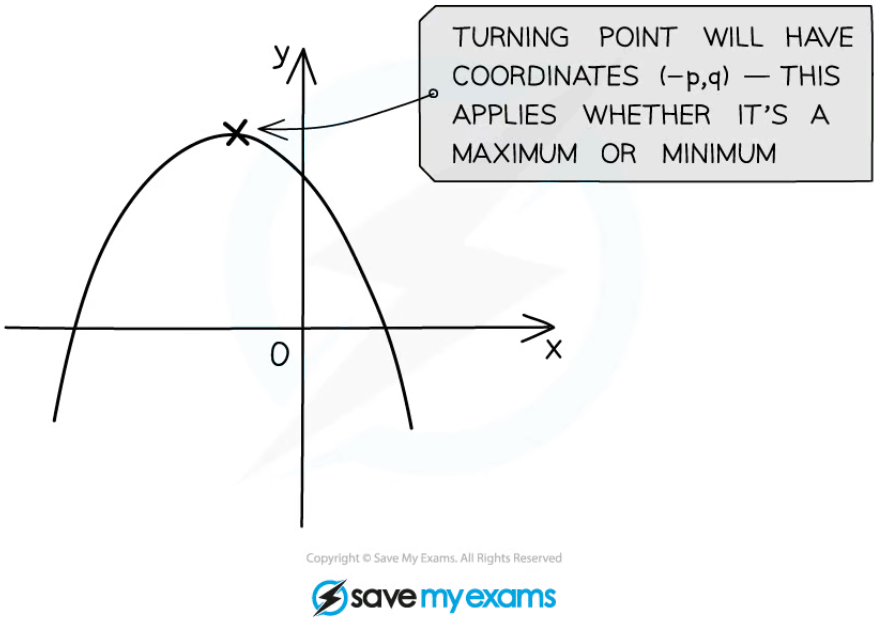
How do I find the turning point by completing the square?

- Completing the square helps us find the **turning point** on a quadratic graph
  - If $y = (x+p)^2 + q$ then the turning point is at $(-p, q)$
    - Notice the negative sign in the *x*-coordinate
    - This links to transformations of graphs (translating $y = x^2$ by *p* to the left and *q* up)
  - If $y = a(x+p)^2 + q$ then the turning point is still at $(-p, q)$
    - It's at a minimum point if *a* > 0
    - It's at a maximum point if *a* < 0
- It can also help you **create the equation** of a quadratic when given the turning point

Your notes

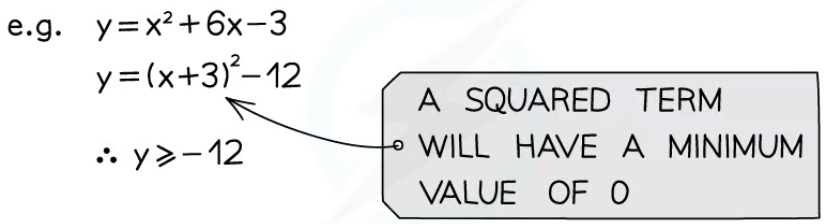
© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



Copyright © Save My Exams. All Rights Reserved
save my exams

- It can also be used to **prove** and/or **show** results using the fact that any "squared term", i.e. the bracket $(x \pm p)^2$, will always be greater than or equal to 0
  - You cannot square a number and get a negative value



Copyright © Save My Exams. All Rights Reserved
save my exams

Exam Tip

- To know if you have completed the square correctly, expand your answer to check.

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

(a) By completing the square, find the coordinates of the turning point on the graph of $y = x^2 + 6x - 11$.

Find half of +6 (call this p)

$$p = \frac{6}{2} = 3$$

Write $x^2 + 6x$ in the form $(x + p)^2 - p^2$

$x^2 + 6x$ is the same as $(x + 3)^2 - 3^2$

Put this result into the equation of the curve

$$y = (x + 3)^2 - 3^2 - 11$$

Simplify the numbers

$$y = (x + 3)^2 - 20$$

Use that the turning point of $y = (x + p)^2 + q$ is at $(-p, q)$

$p = 3$ and $q = -20$

turning point at (-3, -20)

(b) Write $-3x^2 + 12x + 24$ in the form $a(x + p)^2 + q$

Factorise -3 out of the first two terms only
Use square-shaped brackets

$$-3[x^2 - 4x] + 24$$

Complete the square on the $x^2 - 4x$ inside the brackets (write in the form $(x + p)^2 - p^2$ where p is half of -4)

Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

$$-3[(x-2)^2-(-2)^2]+24$$

Simplify the numbers inside the brackets
$(-2)^2$ is 4

$$-3[(x-2)^2-4]+24$$

Multiply -3 by all the terms inside the square-shaped brackets

$$-3(x-2)^2+12+24$$

Simplify the numbers

$$-3(x-2)^2+36$$

This is now in the form $a(x+p)^2+q$ where $a = -3$, $p = -2$ and $q = 36$

$$\mathbf{-3(x-2)^2+36}$$

Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Solving by Completing the Square

Your notes

How do I solve a quadratic equation by completing the square?

- To solve $x^2 + bx + c = 0$
  - **replace** the first two terms, $\boldsymbol{x^2 + bx}$, with $\boldsymbol{(x + p)^2 - p^2}$ where ***p* is half of *b***
  - this is called **completing the square**
    - $x^2 + bx + c = 0$ becomes
      - $(x + p)^2 - p^2 + c = 0$ where p is half of b
  - rearrange this equation to **make *x* the subject** (using ±√)
- For example, solve $x^2 + 10x + 9 = 0$ by completing the square
  - $x^2 + 10x$ becomes $(x + 5)^2 - 5^2$
  - so $x^2 + 10x + 9 = 0$ becomes $(x + 5)^2 - 5^2 + 9 = 0$
  - make x the subject (using ±√)
    - $(x + 5)^2 - 25 + 9 = 0$
    - $(x + 5)^2 = 16$
    - $x + 5 = \pm\sqrt{16}$
    - $x = \pm 4 - 5$
    - $x = -1$ or $x = -9$
- If the equation is $ax^2 + bx + c = 0$ with a number in front of x^2, then divide both sides by a first, before completing the square

How does completing the square link to the quadratic formula?

- The quadratic formula actually comes from completing the square to solve $ax^2 + bx + c = 0$
  - a, b and c are left as letters, to be as general as possible
- You can see hints of this when you solve quadratics
  - For example, solving $x^2 + 10x + 9 = 0$
    - by completing the square, $(x + 5)^2 = 16$ so $x = \pm 4 - 5$ (from above)
    - by the quadratic formula, $x = \frac{-10 \pm \sqrt{64}}{2} = -5 \pm \frac{8}{2}$ $= \pm 4 - 5$ (the same structure)

Exam Tip

- When making x the subject to find the solutions at the end, don't expand the squared brackets back out again!
  - Remember to use ±√ to get **two** solutions

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

Solve $2x^2 - 8x - 24 = 0$ by completing the square

Divide both sides by 2 to make the quadratic start with x^2

$$x^2 - 4x - 12 = 0$$

Halve the middle number, -4, to get -2
Replace the first two terms, $x^2 - 4x$, with $(x - 2)^2 - (-2)^2$

$$(x-2)^2 - (-2)^2 - 12 = 0$$

Simplify the numbers

$$(x-2)^2 - 4 - 12 = 0$$
$$(x-2)^2 - 16 = 0$$

Add 16 to both sides

$$(x-2)^2 = 16$$

Square root both sides
Include the ± sign to get two solutions

$$x - 2 = \pm\sqrt{16} = \pm 4$$

Add 2 to both sides

$$x = \pm 4 + 2$$

Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Work out each solution separately

$x = 6$ or $x = -2$


Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Quadratic Equation Methods


Your notes

Quadratic Equation Methods

If you have to solve a quadratic equation but are not told which method to use, here is a guide as to what to do

When should I solve by factorisation?

- When the question asks to **solve by factorisation**
  - For example, part (a) Factorise $6x^2 + 7x - 3$, part (b) Solve $6x^2 + 7x - 3 = 0$
- When solving **two-term quadratic equations**
  - For example, solve $x^2 - 4x = 0$
    - ...by taking out a **common factor** of x to get $x(x - 4) = 0$
    - ...giving $x = 0$ and $x = 4$
  - For example, solve $x^2 - 9 = 0$
    - ...using the **difference of two squares** to factorise it as $(x + 3)(x - 3) = 0$
    - ...giving $x = -3$ and $x = 3$
    - (Or by rearranging to $x^2 = 9$ and using $\pm\sqrt{}$ to get $x = \ = \pm 3$)

When should I use the quadratic formula?

- When the question says to leave solutions correct to a **given accuracy** (2 decimal places, 3 significant figures etc)
- When the quadratic formula may be **faster** than factorising
  - It's quicker to solve $36x^2 + 33x - 20 = 0$ using the quadratic formula then by factorisation
- **If in doubt**, use the quadratic formula - it always works

When should I solve by completing the square?

- When part (a) of a question says to **complete the square** and part (b) says to use part (a) to solve the equation
- When making x the **subject of harder formulae** containing x^2 and x terms
  - For example, make x the subject of the formula $x^2 + 6x = y$
    - Complete the square: $(x + 3)^2 - 9 = y$
    - Add 9 to both sides: $(x + 3)^2 = y + 9$
    - Take square roots and use $\pm$: $x + 3 = \pm\sqrt{y + 9}$
    - Subtract 3: $x = -3 \pm \sqrt{y + 9}$

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Exam Tip

- Calculators can solve quadratic equations so use them to check your solutions
- If the solutions on your calculator are whole numbers or fractions (with no square roots), this means the quadratic equation does factorise



Your notes

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Worked example

(a) Solve $x^2 - 7x + 2 = 0$, giving your answers correct to 2 decimal places

"Correct to 2 decimal places" suggests using the quadratic formula
Substitute $a = 1$, $b = -7$ and $c = 2$ into the formula, putting brackets around any negative numbers

$$x = \frac{-(-7) \pm \sqrt{(-7)^2 - 4 \times 1 \times 2}}{2 \times 1}$$

Use a calculator to find each solution

$x = 6.70156...$ or $0.2984...$

Round your final answers to 2 decimal places

$x = 6.70$ or $x = 0.30$

(b) Solve $16x^2 - 82x + 45 = 0$

Method 1
If you cannot spot the factorisation, use the quadratic formula
Substitute $a = 16$, $b = -82$ and $c = 45$ into the formula, putting brackets around any negative numbers

$$x = \frac{-(-82) \pm \sqrt{(-82)^2 - 4 \times 16 \times 45}}{2 \times 16}$$

Use a calculator to find each solution

$x = \frac{9}{2}$ or $x = \frac{5}{8}$

Method 2
If you do spot the factorisation, $(2x - 9)(8x - 5)$, then use that method instead

$$(2x - 9)(8x - 5) = 0$$

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Set the first bracket equal to zero

$$2x - 9 = 0$$

Add 9 to both sides then divide by 2

$$2x = 9$$
$$x = \frac{9}{2}$$

Set the second bracket equal to zero

$$8x - 5 = 0$$

Add 5 to both sides then divide by 8

$$8x = 5$$
$$x = \frac{5}{8}$$

$x = \frac{9}{2}$ or $x = \frac{5}{8}$

(c) By writing $x^2 + 6x + 5$ in the form $(x + p)^2 + q$, solve $x^2 + 6x + 5 = 0$

This question wants you to complete the square first
Find p (by halving the middle number)

$$p = \frac{6}{2} = 3$$

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Write $x^2 + 6x$ as $(x + p)^2 - p^2$

$$x^2 + 6x = (x + 3)^2 - 3^2$$
$$= (x + 3)^2 - 9$$

Replace $x^2 + 6x$ with $(x + 3)^2 - 9$ in the equation

$$(x + 3)^2 - 9 + 5 = 0$$
$$(x + 3)^2 - 4 = 0$$

Make x the subject of the equation (start by adding 4 to both sides)

$$(x + 3)^2 = 4$$

Take square roots of both sides (include a ± sign to get both solutions)

$$x + 3 = \pm\sqrt{4} = \pm 2$$

Subtract 3 from both sides

$$x = \pm 2 - 3$$

Find each solution separately using + first, then - second

$x = -5, x = -1$

Even though the quadratic factorises to $(x + 5)(x + 1)$, this is not the method asked for in the question

© 2015–2024 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers