CIE A Level Maths: Pure 3

Your notes

7.3 Further Vectors

Contents

7.3.1 Equation of a Line in Vector Form

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Equation of a Line in Vector Form

How do I find the vector equation of a line?

- You need to know:
  - The **position vector** of one point on the line
  - A **direction vector** of the line (or the position vector of another point)
- There are two formulas for getting a **vector equation** of a line:
  - **r** = **a** + *t* (**b** - **a**)
    - use this formula when you know the position vectors **a** and **b** of two points on the line
  - **r** = **a** + *t* **d**
    - use this formula when you know the position vector **a** of a point on the line and a direction vector **d**
  - Both forms could be compared to the Cartesian equation of a 2D line
    - $y = mx + c$
    - The point on the line **a** is similar to the "+c" part
    - The direction vector **d** or **b** – **a** is similar to the "m" part
- The vector equation of a line shown above can be applied equally well to vectors in **2 dimensions** and to vectors in **3 dimensions**
- Recall that vectors may be written using i, j, k **reference unit vectors** or as **column vectors**
- It follows that in a vector equation of a line either form can be employed – for example,

$$r = 3i + j - 7k + t(i - 2j) \text{ and } r = \begin{pmatrix} 3 \\ 1 \\ -7 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \\ 0 \end{pmatrix}$$

show the same equation written using the two different forms

How do I determine if a point is on a line?

- Each **different point** on the line corresponds to a **different value of** *t*
  - For example: if an equation for a line is **r** = 3**i** + 2**j** - **k** + t (**i** + 2**j**)
    - the point with coordinates (2, 0, -1) is on the line and corresponds to *t* = *-1*
  - However we know that the point with coordinates (-7, 5, 0) is not on this line
    - No value of *t* could make the **k** component 0

Can two different equations represent the same line?

- Why do we say *a* direction vector and not *the* direction vector? Because the magnitude of the vector doesn't matter; **only the direction is important**
  - we can multiply any direction vector by a (non-zero) constant and this wouldn't change the direction
- Therefore there are an infinite number of options for **a** (a point on the line) and an infinite number of options for the direction vector
- For Cartesian equations – two equations will represent the same line only if they are multiples of each other
  - $x - 2y = 5$ and $3x - 6y = 15$
- For vector equations this is not true – two equations might look different but still represent the same line:
  - $\mathbf{r} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} + t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -2 \end{pmatrix} + t\begin{pmatrix} -2 \\ -1 \end{pmatrix}$

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Worked example

a) Find a vector equation of the line through the points $A\ (1, -8)$ and $B\ (-3, 0)$.

b) Find a vector equation of the line through $(4, 0, -5)$ in the direction $\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$.

a) WE ARE GIVEN TWO POINTS, AND SO WE MUST USE THE FORMULA

$$r = a + t(b - a)$$

$$a = \begin{pmatrix} 1 \\ -8 \end{pmatrix} \quad b = \begin{pmatrix} -3 \\ 0 \end{pmatrix}$$

THESE ARE THE POSITION VECTORS OF THE POINTS.

SUBTRACT THE POSITION VECTORS (THE ORDER DOESN'T MATTER) TO GET A DIRECTION VECTOR FOR THE LINE

$$r = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t\left(\begin{pmatrix} -3 \\ 0 \end{pmatrix} - \begin{pmatrix} 1 \\ -8 \end{pmatrix}\right)$$

SO

$$r = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t\begin{pmatrix} -4 \\ 8 \end{pmatrix}$$

YOU CAN ALSO WRITE THIS AS

$$r = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

MOVING IN THE DIRECTION '4 LEFT AND 8 UP' IS THE SAME AS MOVING IN THE DIRECTION '1 LEFT AND 2 UP'

OR EVEN AS

$$r = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} -1 \\ 2 \end{pmatrix}$$

YOU CAN USE ANY POINT ON THE LINE

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b) WE ARE GIVEN A POINT AND A DIRECTION VECTOR, AND SO WE USE THE FORMULA

$$r = a + td$$

SINCE THE COORDINATES OF THE GIVEN POINT ON THE LINE ARE (4, 0, −5) WE HAVE $A = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix}$

A DIRECTION VECTOR IS $d = \begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$

$$r = \begin{pmatrix} 4 \\ 0 \\ -5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 2 \end{pmatrix}$$

Exam Tip

- Remember that the vector equation of a line can take many different forms. This means that the answer you derive might look different from the answer in a mark scheme.
- You can choose whether to write your vector equations of lines using reference unit vectors or as column vectors – use the form that you prefer!
- If, for example, an exam question uses column vectors, then it is usual to leave the answer in column vectors, but it isn't essential to do so - you'll still get the marks!

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7.3.2 Parallel, Intersecting & Skew Lines

Parallel, Intersecting & Skew Lines

In two dimensions, lines are either parallel or they intersect at a single point. If they are parallel, then either they have no points in common, or they share every point in common.

In three dimensions, there is a further possibility: a pair of lines might not be parallel and have no points of intersection. We say that the lines are skew.

How do I tell if two lines are parallel?

- Two lines are parallel if, and only if, their **direction vectors** are **parallel**
  - This means the direction vectors will be **scalar multiples** of each other
  - For example, the lines whose equations are $\mathbf{r} = \begin{pmatrix} 2 \\ 1 \\ -7 \end{pmatrix} + s\begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} 1 \\ -1 \\ 5 \end{pmatrix} + t\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$ are parallel since their direction vectors $\begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix}$ and $\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$ are parallel vectors as $\begin{pmatrix} 2 \\ 0 \\ -8 \end{pmatrix} = -2\begin{pmatrix} -1 \\ 0 \\ 4 \end{pmatrix}$
- There are two possibilities for two parallel lines: either they **never intersect** or they are the **identical**
  - Recall that the vector equation of a line can take **many forms** – for example, the lines represented by the equations $\mathbf{r} = \begin{pmatrix} 1 \\ -8 \end{pmatrix} + s\begin{pmatrix} -4 \\ 8 \end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix} -3 \\ 0 \end{pmatrix} + t\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ are actually the same line even though the equations look entirely different
- To see that the lines are identical, first check that they are **parallel**
  - they are because $\begin{pmatrix} -4 \\ 8 \end{pmatrix} = -4\begin{pmatrix} 1 \\ -2 \end{pmatrix}$ and so the direction vectors are parallel

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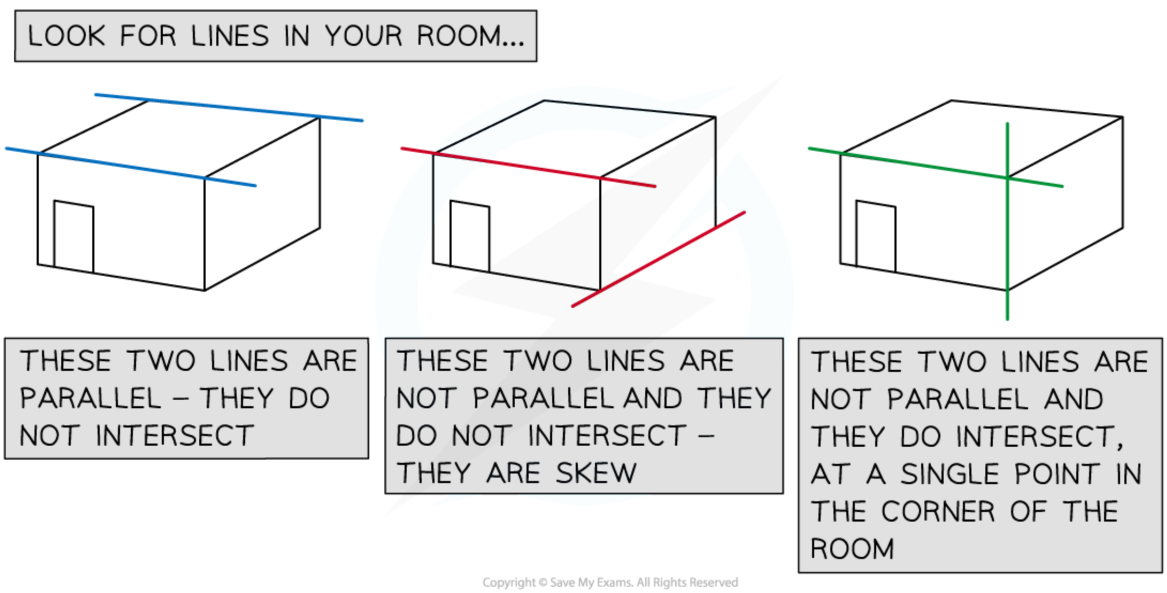
- Next, determine whether **any point** on one of the lines also lies on the other.
  - In the example above, is the position vector of a point on the first line – does it also lie on the second line? Yes, because $\begin{pmatrix}1\\-8\end{pmatrix}=\begin{pmatrix}-3\\0\end{pmatrix}+4\begin{pmatrix}1\\-2\end{pmatrix}$
- If two parallel lines share **any point**, then they share **all points** – i.e. they are **identical**

What are skew lines?

- First, start with another question: do lines which are not parallel necessarily intersect?
  - In **2 dimensions**, the answer is **yes**
  - However, lines in **3 dimensions do not necessarily intersect**
- Lines that are **not parallel** and which **do not intersect** are called **skew lines**



How do I determine whether lines in 3D are parallel, skew, or intersecting?

- First, look to see if the direction vectors are parallel:
  - if the **direction vectors are parallel**, then the **lines are parallel**
  - if the **direction vectors are not parallel**, the **lines are not parallel**
- If the lines are **parallel**, check to see if they are **identical**:
  - If they **share any point**, then they are **identical**
  - If **any point** on one line is **not on the other line**, then the lines are **not identical**

- If the lines are **not parallel**, check whether they intersect:
  - Using **different letters,** e.g. *s* and *t*, for the parameters, write down coordinates for a general point on each line
  - Supposing that the lines do intersect: equate the two coordinates and write down **three equations**
    - One for each component (**i, j, k**)
  - Solve **any two pairs** of these equations simultaneously to find *s* and *t*
  - **Check** whether the values of *s* and *t* you have found satisfy the third equation
    - If **all three** equations are satisfied, then the lines **intersect**
    - If **not all three** equations are satisfied, then the lines are **skew**
- If a pair of lines are **not parallel** and **do intersect**, the unique point of intersection can be found by substituting the value of one of the parameters you have found into the coordinates for points on the appropriate line.

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Worked example

Determine whether the following pair of lines are parallel, intersect, or are skew:

$$r = 4i + 3j + s(5i + 2j + 3k) \text{ and}$$
$$r = -5i + 4j + k + t(2i - j).$$

FIRST WE HAVE TO CHECK WHETHER THE LINES ARE PARALLEL. TO DO THIS, WE LOOK TO SEE WHETHER THEIR DIRECTION VECTORS ARE PARALLEL.

THESE LINES ARE NOT PARALLEL BECAUSE THE DIRECTION VECTORS $5i + 2j + 3k$ AND $2i - j$ ARE NOT MULTIPLES OF EACH OTHER.

WE HAVE TO DETERMINE WHETHER THEY ARE SKEW.

ANY POINT ON THE FIRST LINE HAS COORDINATES $(4 + 5s, 3 + 2s, 3s)$ AND ANY POINT ON THE SECOND LINE HAS COORDINATES $(-5 + 2t, 4 - t, 1)$.

WE SUPPOSE THAT THE LINES DO MEET AND WRITE DOWN THE THREE EQUATIONS OBTAINED BY EQUATING THE COORDINATES.

IF THE LINES DO INTERSECT, THEN IT WILL BE POSSIBLE TO FIND A VALUE FOR s AND a VALUE FOR t THAT ALL THREE OF THE EQUATIONS BELOW ARE SATISFIED. IF THE LINES ARE SKEW, IT WILL BE POSSIBLE TO SOLVE ONLY TWO OF THE EQUATIONS.

$4 + 5s = -5 + 2t$ (A)
$3 + 2s = 4 - t$ (B)
$3s = 1$ (C)

LABEL THE EQUATIONS – IT HELPS YOU TO EXPLAIN WHAT YOU'RE DOING.

WE SOLVE ANY TWO OF THE THREE EQUATIONS (A), (B) AND (C) SIMULTANEOUSLY – PICK THE EASIEST TWO!

$(C) \Rightarrow \quad s = \frac{1}{3}$

SUBSTITUTE $s = \frac{1}{3}$ INTO (B)

$(B) \Rightarrow \quad 3 + 2 \times \frac{1}{3} = 4 - t$

$\Rightarrow \quad \frac{11}{3} = 4 - t$

$\Rightarrow \quad t = 4 - \frac{11}{3} = \frac{1}{3}$

NOW SUBSTITUTE $s = \frac{1}{3}$ AND $t = \frac{1}{3}$ INTO (A):

$(A) \Rightarrow \quad 4 + 5 \times \frac{1}{3} = -5 + 2 \times \frac{1}{3}$

$\Rightarrow \quad \frac{17}{3} = -\frac{13}{3}$

THIS IS A CONTRADICTION, AND SO THE LINES ARE SKEW.

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Exam Tip

- Make sure that you use different letters, e.g. *s* and *t*, to represent the parameters in vector equations of different lines.

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7.3.3 The Scalar ('Dot') Product

The Scalar ('Dot') Product

The **scalar product** is an important link between the algebra of vectors and the trigonometry of vectors. We shall see that the scalar product is somewhat comparable to the operation of multiplication on real numbers.

What is the scalar (dot) product?

- The scalar product between two vectors **a** and **b** is represented by $\mathbf{a} \cdot \mathbf{b}$
  - This is also called the dot product because of the symbol used
- The scalar product between two vectors $\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$ and $\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$ is defined as $\mathbf{a} \cdot \mathbf{b} = a_1b_1 + a_2b_2 + a_3b_3$
- The result of taking the scalar product of two vectors is a **real number**
  - i.e. a **scalar**
- For example,

$$(3\mathbf{i} - \mathbf{k}) \cdot (2\mathbf{i} + 9\mathbf{j} + \mathbf{k}) = 3 \times 2 + 0 \times 9 + (-1) \times 1 = 6 + 0 - 1 = 5$$

and

$$\begin{pmatrix} 2 \\ 7 \end{pmatrix} \cdot \begin{pmatrix} -8 \\ 2 \end{pmatrix} = 2 \times (-8) + 7 \times 2 = -16 + 14 = -2$$

- The scalar product has some important properties:
  - The order of the vectors doesn't affect the result:

$$\mathbf{a} \cdot \mathbf{b} = \mathbf{b} \cdot \mathbf{a}$$

- In effect we can 'multiply out' brackets:

$$\mathbf{a} \cdot (\mathbf{b} + \mathbf{c}) = \mathbf{a} \cdot \mathbf{b} + \mathbf{a} \cdot \mathbf{c}$$

- This means that we can do many of the same things with vectors as we can do when operating on real numbers – for example,

$$(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b}) = \mathbf{a} \cdot \mathbf{a} - 2\mathbf{a} \cdot \mathbf{b} + \mathbf{b} \cdot \mathbf{b}$$

- The scalar product between a vector and itself is equal to the square of its **magnitude:**

$$\mathbf{a} \cdot \mathbf{a} = |\mathbf{a}|^2$$

For example,

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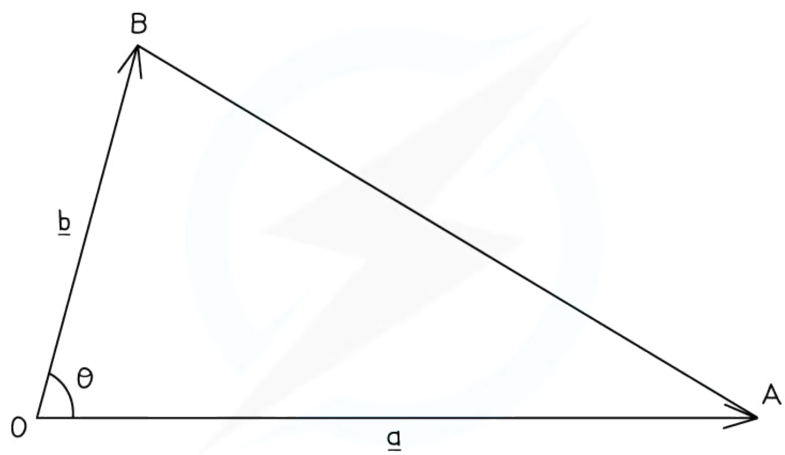
$$\begin{pmatrix}2\\7\end{pmatrix}\cdot\begin{pmatrix}2\\7\end{pmatrix} = 2^2 + 7^2 = 53 \text{ and } \left|\begin{pmatrix}2\\7\end{pmatrix}\right|^2 = 2^2 + 7^2 = 53$$

What is the connection between the scalar product and trigonometry?

- There is another important method for finding $\mathbf{a} \cdot \mathbf{b}$ involving the angle between the two vectors θ:

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\,\theta$$

  - Here θ is the angle between the vectors when they are placed **'base to base'**
    - when the vectors are placed so that they begin at the same point
  - This formula can be derived using the cosine rule and expanding $(\mathbf{a} - \mathbf{b}) \cdot (\mathbf{a} - \mathbf{b})$



Worked example

In each case, calculate the scalar product $a \bullet b$ between the pairs of vectors:

(a) $a = 2i + j - 7k, \quad b = 3i + 5j - k;$

(b) $a = 3i - 4j - 12k$, and b is a vector of magnitude 1. The angle between the vectors is 60°.

a) USE THE FORMULA $a \cdot b = a_1 b_1 + a_2 b_2 + a_3 b_3$

$$a \cdot b = 2 \times 3 + 1 \times 5 + (-7) \times (-1)$$
$$= 6 + 5 + 7$$
$$= 18$$

b) FIRST, WE NEED TO CALCULATE $|a|$.

$$|a| = \sqrt{3^2 + 4^2 + 12^2} = \sqrt{9 + 16 + 144} = 13$$

WE CAN IGNORE THE SIGN OF THE COMPONENTS – ONLY THE SIZE MATTERS

NEXT USE THE FORMULA $a \cdot b = |a|\,|b| \cos\theta$.

$$a \cdot b = 13 \times 1 \times \cos 60°$$
$$= 13 \times 1 \times \frac{1}{2}$$
$$= 6.5$$

Exam Tip

- When writing a scalar product, it's important to write a distinctive **dot** between the vectors – otherwise your meaning will not be clear.

Your notes

Your notes

7.3.4 Uses of the Scalar Product

Uses of the Scalar Product

This revision note covers several applications of the scalar product for vectors – namely, how you can use the scalar product to:

- find the angle between vectors or lines
- test whether vectors or lines are perpendicular
- find the closest distance from a point to a line

How do I find the angle between two vectors?

- Recall that a formula for the scalar (or 'dot') between vectors $\mathbf{a}$ and $\mathbf{b}$ is

$$\mathbf{a}\cdot\mathbf{b} = |\mathbf{a}||\mathbf{b}|\cos\theta$$

  - where θ is the angle between the vectors when they are placed **'base to base'**
    - that is, when the vectors are positioned so that they start at the same point
  - We arrange this formula to make $\cos\,\theta$ the subject:
  - To find the angle between two vectors
    - Calculate the scalar product between them
    - Calculate the magnitude of each vector
    - Use the formula to find $\cos\,\theta$
    - Use inverse trig to find θ

How do I find the angle between two lines?

- To find the angle between two lines, find the angle between their **direction vectors**
  - For example, if the lines have equations $\mathbf{r} = \mathbf{a}_1 + s\mathbf{d}_1$ and $\mathbf{r} = \mathbf{a}_2 + t\mathbf{d}_2$, then the angle θ between the lines is given by

$$\theta = \cos^{-1}\left(\frac{\mathbf{d}_1\cdot\mathbf{d}_2}{|\mathbf{d}_1||\mathbf{d}_2|}\right)$$

How do I tell if vectors or lines are perpendicular?

- Two (non-zero) vectors $\mathbf{a}$ and $\mathbf{b}$ are **perpendicular** if, and only if, $\mathbf{a}\cdot\mathbf{b} = 0$
  - If the **a** and **b** are perpendicular then:

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- $\theta = 90° \Rightarrow \cos\theta = 0 \Rightarrow |\mathbf{a}||\mathbf{b}|\cos\theta = 0 \Rightarrow \mathbf{a}\cdot\mathbf{b} = 0$
- If $\mathbf{a}\cdot\mathbf{b} = 0$ then:
  - $|\mathbf{a}||\mathbf{b}|\cos\theta = 0 \Rightarrow \cos\theta = 0 \Rightarrow \theta = 90° \Rightarrow$ **a** and **b** are perpendicular
- For example, the vectors $2i - 3j + 5k$ and $-4i - j + k$ are perpendicular since

$$(2i - 3j + 5k)\cdot(-4i - j + k) = 2\times(-4) + (-3)\times(-1) + 5\times 1 = -8 + 3 + 5 = 0$$

How do I find the shortest distance from a point to a line?

- Suppose that we have a line l with equation $\mathbf{r} = \mathbf{a} + t\mathbf{d}$ and a point P not on l
- Let F be the **point on** l which is **closest to** $\boldsymbol{P}$ (sometimes called the **foot of the perpendicular**)
  - Then the line between F and P will be perpendicular to the line l
- To find the closest point F
  - Call $\mathbf{f} = \overrightarrow{\mathrm{OF}}$ and $\mathbf{p} = \overrightarrow{OP}$
  - Since F lies on l, we have $\mathbf{f} = \mathbf{a} + t_0\mathbf{d}$, for a unique real number t_0
  - Find the vector $\overrightarrow{\boldsymbol{FP}}$ using $\mathbf{p} - \mathbf{f}$
  - $\overrightarrow{\boldsymbol{FP}}$ **is perpendicular to** $\boldsymbol{d}$ so form an equation using $(\mathbf{p} - \mathbf{f})\cdot\mathbf{d} = 0$
  - Solve this equation to find the value of t_0
  - Use the value of t_0 to find $\mathbf{f}$
- The **shortest distance** between the point and the line is the **length**
- Note that the **shortest distance between the point and the line** is sometimes referred to as the **length of the perpendicular**

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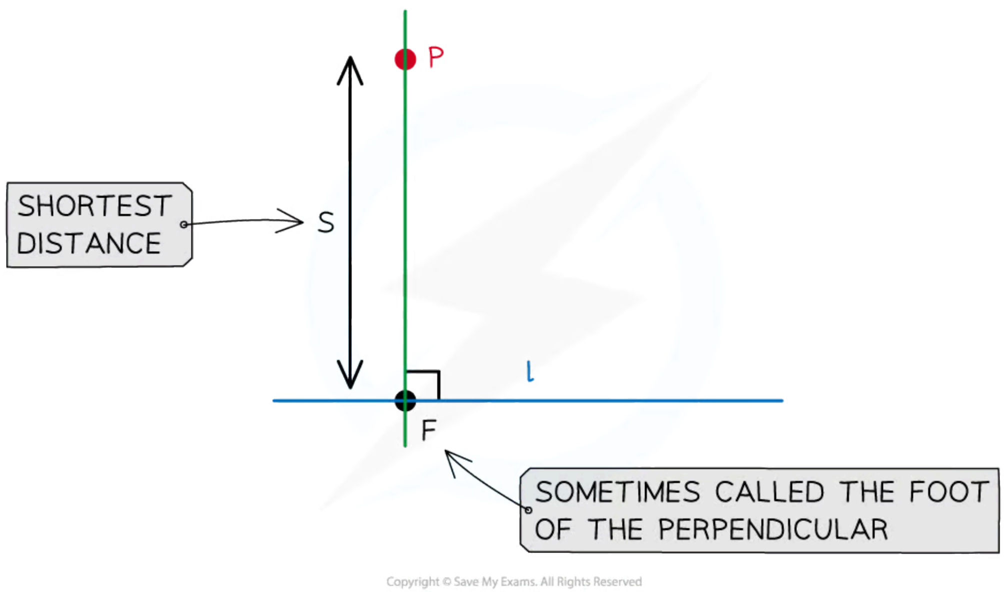
P
SHORTEST DISTANCE
S
l
F
SOMETIMES CALLED THE FOOT OF THE PERPENDICULAR
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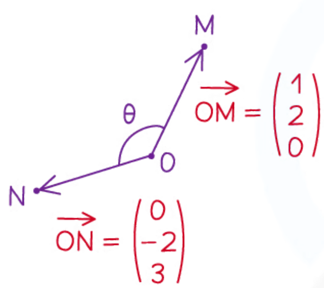
Worked example

The points M and N have coordinates $(1,2,0)$ and $(0,-2,3)$, respectively.

(a) Calculate the angle between the two vectors $\overrightarrow{OM}$ and $\overrightarrow{ON}$.

(b) Show that the shortest distance between M and the line $\mathbf{r} = \begin{pmatrix} 2 \\ 0 \\ 6 \end{pmatrix} + t\begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}$ is $\sqrt{21}$.

a)



LET $\underline{a} = \overrightarrow{OM}$ AND $\underline{b} = \overrightarrow{ON}$

CALCULATE $a \cdot b$, $|a|$ AND $|b|$

$\underline{a} \cdot \underline{b} = 1 \times 0 + 2 \times (-2) + 0 \times 3 = -4$

$|\underline{a}| = \sqrt{1^2 + 2^2 + 0^2} = \sqrt{5}$

$|\underline{b}| = \sqrt{0^2 + (-2)^2 + 3^2} = \sqrt{13}$

USE $\cos\theta = \dfrac{\underline{a} \cdot \underline{b}}{|\underline{a}|\,|\underline{b}|}$

$$\cos\theta = \frac{-4}{\sqrt{5} \times \sqrt{13}}$$

$$\theta = \cos^{-1}\left(\frac{-4}{\sqrt{5} \times \sqrt{13}}\right) = 119.744\ldots$$

$\theta = 119.7^\circ$ (1dp)

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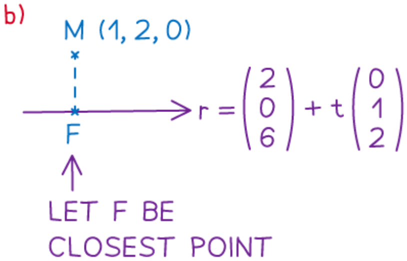
b)



F IS ON THE LINE SO F CAN BE WRITTEN

$$\overrightarrow{OF} = \begin{pmatrix}2\\0\\6\end{pmatrix} + t_0\begin{pmatrix}0\\1\\2\end{pmatrix} = \begin{pmatrix}2\\t_0\\6+2t_0\end{pmatrix}$$

FIND $\overrightarrow{MF}$ USING $\overrightarrow{OF} - \overrightarrow{OM}$

$$\overrightarrow{MF} = \begin{pmatrix}2\\t_0\\6+t_0\end{pmatrix} - \begin{pmatrix}1\\2\\0\end{pmatrix} = \begin{pmatrix}-1\\t_0-2\\6+2t_0\end{pmatrix}$$

$\overrightarrow{MF}$ IS PERPENDICULAR TO DIRECTION OF LINE $\begin{pmatrix}0\\1\\2\end{pmatrix}$

SO $\overrightarrow{MF} \cdot \begin{pmatrix}0\\1\\2\end{pmatrix} = 0$

$$\begin{pmatrix}-1\\t_0-2\\6+2t_0\end{pmatrix} \cdot \begin{pmatrix}0\\1\\2\end{pmatrix} = 0$$

USE SCALAR PRODUCT

$1\times 0 + (t_0 - 2)\times 1 + (6 + 2t_0)\times 2 = 0$

SIMPLIFY

$5t_0 + 10 = 0$

SOLVE

$t_0 = -2$

SUBSTITUTE BACK INTO $\overrightarrow{MF}$

$$\overrightarrow{MF} = \begin{pmatrix}-1\\(-2)-2\\6+2(-2)\end{pmatrix} = \begin{pmatrix}1\\-4\\2\end{pmatrix}$$

FIND THE MAGNITUDE OF $\overrightarrow{MF}$ TO GET SHORTEST DISTANCE

$$|\overrightarrow{MF}| = \sqrt{1^2 + (-4)^2 + 2^2} = \sqrt{21}$$

Exam Tip

It can be easier and clearer to work with column vectors when dealing with scalar products.