



CIE A Level Maths: Pure 3



Your notes

8.2 Argand Diagrams

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Your notes

8.2.1 Argand Diagrams – Basics

Argand Diagrams – Basics

What is an Argand diagram?

- An Argand diagram is a **geometrical** way to represent complex numbers as either a **point** or a **vector** in two-dimensional space
 - We can represent the complex number $x + yi$ by the **point** with **cartesian coordinate** (x, y)
- The **real** component is represented by points on the **x-axis**, called the **real axis**, Re
- The **imaginary** component is represented by points on the **y-axis**, called the **imaginary axis**, Im

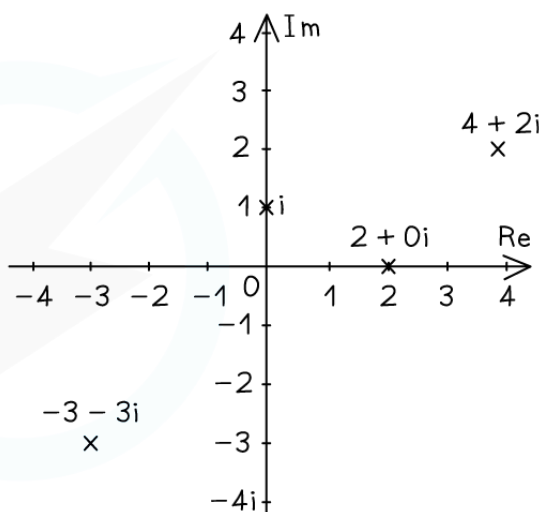
REPRESENTING AS A POINT

$4 + 2i$ REPRESENTED BY $(4, 2)$

$-3 - 3i$ REPRESENTED BY $(-3, -3)$

2 REPRESENTED BY $(2, 0)$

i REPRESENTED BY $(0, 1)$



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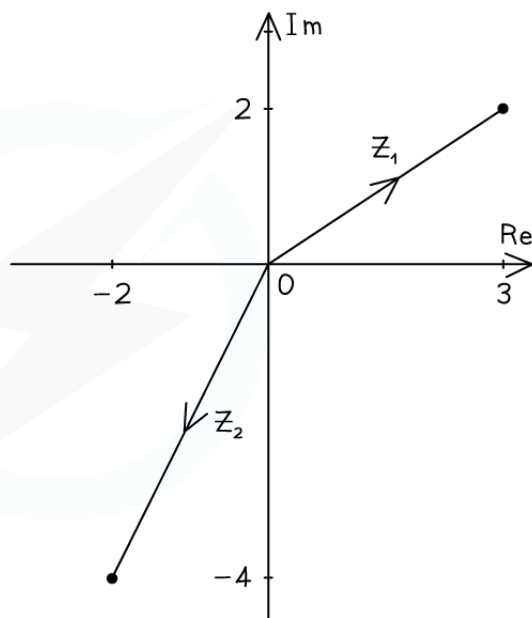


Your notes

REPRESENTING IN POSITION VECTOR FORM

$$z_1 = 3 + 2i$$

$$z_2 = -2 - 4i$$



IN A SKETCH YOU ONLY SHOW THE KEY POINTS ON THE AXES!

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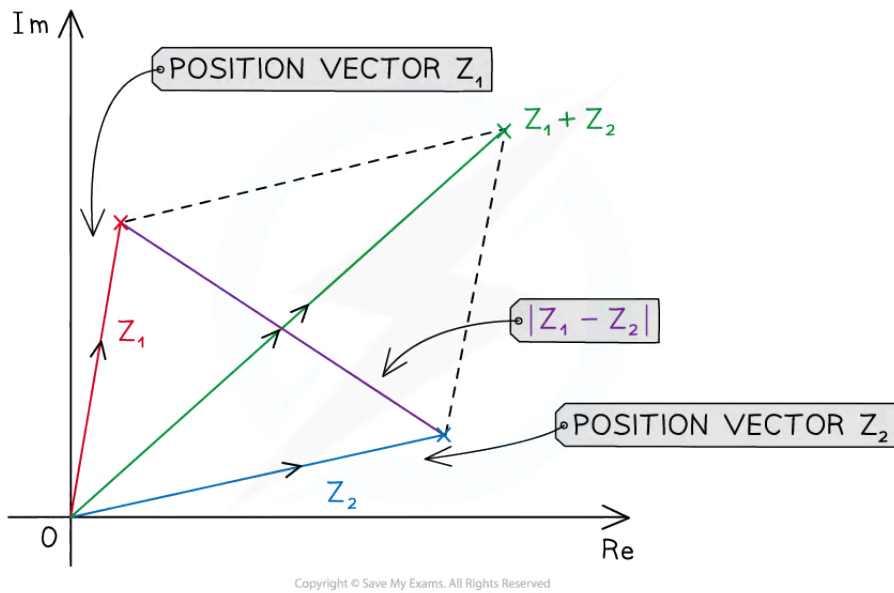
- You may be asked to show **roots** of an equation in an Argand diagram
 - First **solve** the equation
 - Draw a quick sketch, only adding essential information to the axes
 - Plot the points and label clearly

How can I use an Argand diagram to visualise $|z_1 + z_2|$ and $|z_1 - z_2|$?

- Plot two complex numbers z_1 and z_2
- Draw a line from the origin to each complex number
- Form a parallelogram using the two lines as two adjacent sides
- The modulus of their sum $|z_1 + z_2|$ will be the length of the diagonal of the parallelogram starting at the origin
- The modulus of their difference $|z_1 - z_2|$ will be the length of the diagonal between the two complex numbers



Your notes





Your notes

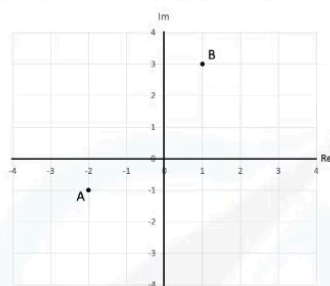
Worked example



$$z_1 = 2 + 2i \text{ and } z_2 = 3 - 4i$$

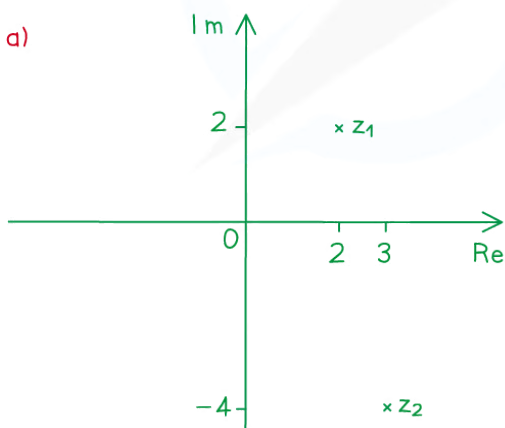
(a) Show the complex numbers z_1 and z_2 as points on an Argand diagram.

(b) On the Argand diagram below two points, A and B, have been plotted. State the complex numbers represented by points A and B.



$$z_1 = 2 + 2i; \quad z_2 = 3 - 4i$$

a)



SHOWN AS POINTS

b)

$$A = -2 - i$$

$$B = 1 + 3i$$

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Examiner Tip

- When setting up an Argand diagram you do not need to draw a fully scaled axes, you only need the essential information for the points you want to show, this will save a lot of time.



Your notes

8.2.2 Geometry of Complex Addition, Subtraction & Conjugation

Geometry of Complex Addition, Subtraction & Conjugation

What does addition look like on an Argand diagram?

- The **addition** of complex numbers can be shown by the addition of **corresponding column vectors**

- If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 + z_2 = (a + bi) + (c + di)$

- This can be written as

$$\begin{pmatrix} a \\ b \end{pmatrix} + \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a + c \\ b + d \end{pmatrix}$$

- An alternative is to write $(a + bi) + (c + di)$ as $(a + c) + (b + d)i$, adding the respective real and imaginary parts separately

- A complex number $x + yi$ can be represented by the **position vector** $\begin{pmatrix} x \\ y \end{pmatrix}$

What does subtraction look like on an Argand diagram?

- As with addition we can use knowledge of vectors to represent **subtraction** of complex numbers

- If $z_1 = a + bi$ and $z_2 = c + di$, then $z_1 - z_2 = (a + bi) - (c + di)$

- This can be written as

$$\begin{pmatrix} a \\ b \end{pmatrix} - \begin{pmatrix} c \\ d \end{pmatrix} = \begin{pmatrix} a - c \\ b - d \end{pmatrix}$$

- An alternative is to write $(a + bi) - (c + di)$ as $(a - c) + (b - d)i$, subtracting the respective real and imaginary parts separately

What are the geometric representations of complex addition and subtraction?

- Let w be a given complex number with **real part a** and **imaginary part b**

- $w = a + bi$

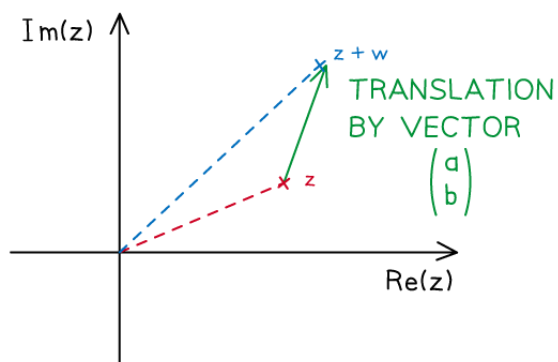
- Let z be any complex number represented on an Argand diagram

- Adding w to z** results in z being **translated** by vector $\begin{pmatrix} a \\ b \end{pmatrix}$

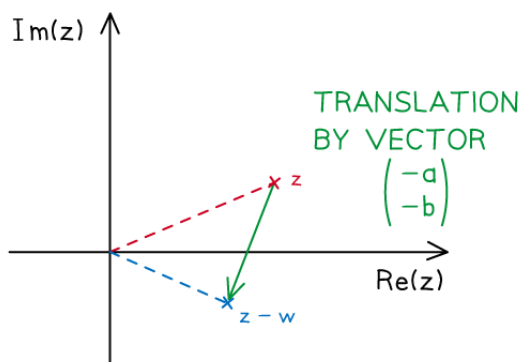
- Subtracting w from z** results in z being **translated** by vector $\begin{pmatrix} -a \\ -b \end{pmatrix}$



Your notes



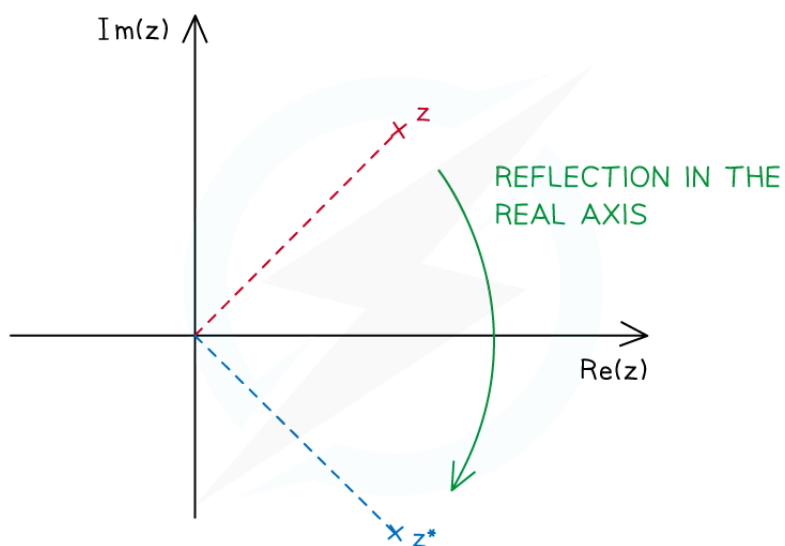
ADDITION OF $w = a + bi$



SUBTRACTION BY $w = a + bi$

What is the geometric representation of complex conjugation?

- If we plot **complex conjugate** pairs on an Argand diagram, we notice the points are **reflections** of each other in the **real axis**
- Let z be any complex number represented on an Argand diagram
- **Complex conjugating z** results in z being **reflected** in the real axis



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 Worked example

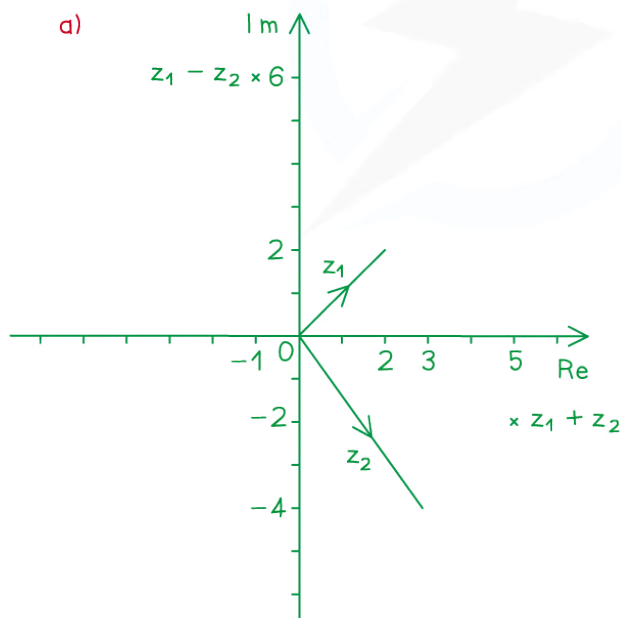


$$z_1 = 2 + 2i \text{ and } z_2 = 3 - 4i$$

- Show the complex numbers z_1 and z_2 as position vectors on an Argand diagram
- Work out the values of $z_1 + z_2$ and $z_1 - z_2$ and plot these complex numbers on the same diagram as in part (a)
- Explain, in relation to the complex numbers z_1 and z_2 , what geometrical effect has occurred when we plot their conjugates

$$z_1 = 2 + 2i; \quad z_2 = 3 - 4i$$

a)



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b) $z_1 + z_2$ $z_1 - z_2$
 $(2 + 2i) + (3 - 4i)$ $(2 + 2i) - (3 - 4i)$
 GROUP BY Re AND Im PARTS
 $(2 + 3) + (2 - 4)i$ $(2 - 3) + (2 - -4)i$
 $(5) + (-2)i$ $(-1) + (6)i$
 $5 - 2i$ $-1 + 6i$
 POINTS SHOWN IN DIAGRAM ABOVE

c) z_1 AND z_2 ARE REFLECTED IN THE REAL AXIS

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Examiner Tip

Read questions carefully; is it asking to plot the complex number as a point or as a vector?

Be extra careful when representing subtraction geometrically, remember that the solution will be a translation of the shorter diagonal of the parallelogram made up by the two vectors.



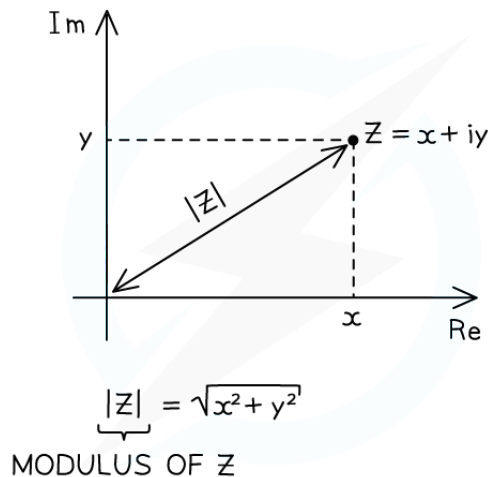
Your notes

8.2.3 Modulus & Argument

Modulus & Argument

How do I find the modulus of a complex number?

- The modulus of a complex number is its **distance** from the origin when plotted on an Argand diagram
- The modulus of Z is written $|Z|$
- If $Z = x + iy$, then we can use **Pythagoras** to show...
 - $|Z| = \sqrt{x^2 + y^2}$
- A modulus is always **positive**
- the modulus is related to the complex **conjugate** by...
 - $ZZ^* = Z^*Z = |Z|^2$
 - This is because $ZZ^* = (x + iy)(x - iy) = x^2 + y^2$
- In general, $|Z_1 + Z_2| \neq |Z_1| + |Z_2|$
 - e.g. both $Z_1 = 3 + 4i$ and $Z_2 = -3 + 4i$ have a modulus of 5, but $Z_1 + Z_2$ simplifies to $8i$ which has a modulus of 8



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How do I find the argument of a complex number?

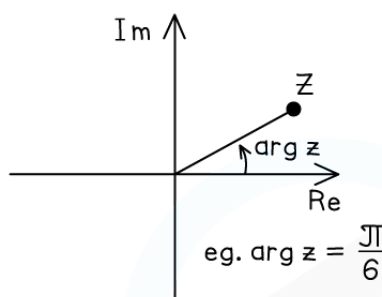
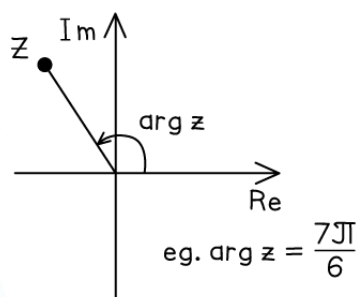
- The argument of a complex number is the anti-clockwise **angle** that it makes when starting at the positive real axis on an Argand diagram
- Arguments are measured in **radians**
 - Sometimes these can be given exact in terms of π
- The argument of Z is written **arg Z**
- Arguments can be calculated using right-angled **trigonometry**

- This involves using the tan ratio plus a sketch to decide whether it is positive/negative and acute/obtuse
- Arguments are usually given in the range $-\pi < \arg z \leq \pi$
 - Negative arguments are for complex numbers in the third and fourth quadrants
 - Occasionally you could be asked to give arguments in the range $0 \leq \arg z < 2\pi$
- The argument of zero, $\arg 0$ is undefined (no angle can be drawn)

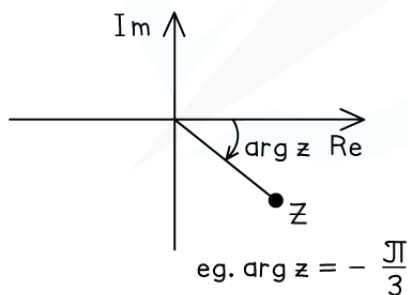
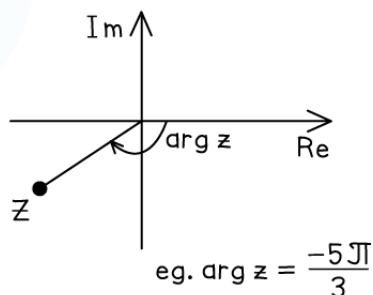


Your notes

POSITIVE ARGUMENTS



NEGATIVE ARGUMENTS



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Your notes

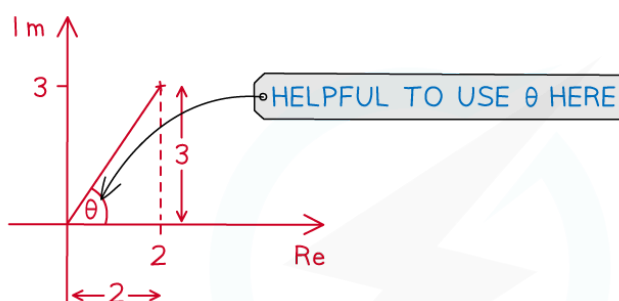
 Worked example



(a) Find the modulus and argument of $z = 2 + 3i$

(b) Find the modulus and argument of $w = -1 - \sqrt{3}i$

d) **DRAW A SKETCH TO HELP**



USING PYTHAGORAS

$$|z| = \sqrt{2^2 + 3^2} = (\sqrt{13})$$

$$= 3.6055...$$

USING TRIGONOMETRY

$$\tan \theta = \frac{3}{2}$$

$$\theta = \tan^{-1} \left(\frac{3}{2} \right)$$

$$= 0.98279...$$

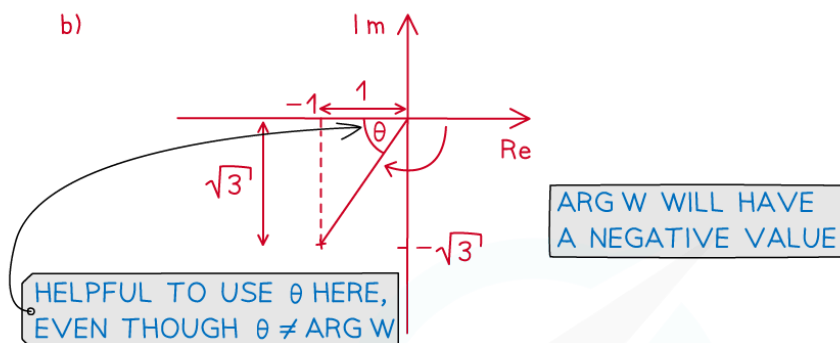
$$|z| = \sqrt{13} = 3.61 \text{ TO 3sf AND } \arg z = 0.983 \text{ RADIANS TO 3sf}$$

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b)



$$|w| = \sqrt{(-1)^2 + (-\sqrt{3})^2} (= \sqrt{4})$$

$$= 2$$

$$\tan \theta = \frac{\sqrt{3}}{1}$$

$$\theta = \tan^{-1}(\sqrt{3})$$

$$= \frac{\pi}{3}$$

NEED OBTUSE ANGLE:

$$\pi - \frac{\pi}{3} = \frac{2\pi}{3}$$

$$|w| = 2 \quad \text{ARG } w = -\frac{2\pi}{3} = -2.09 \text{ TO 3sf}$$

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Examiner Tip

- Give non-exact arguments in radians to 3 significant figures.



Your notes

Modulus-Argument (Polar) Form

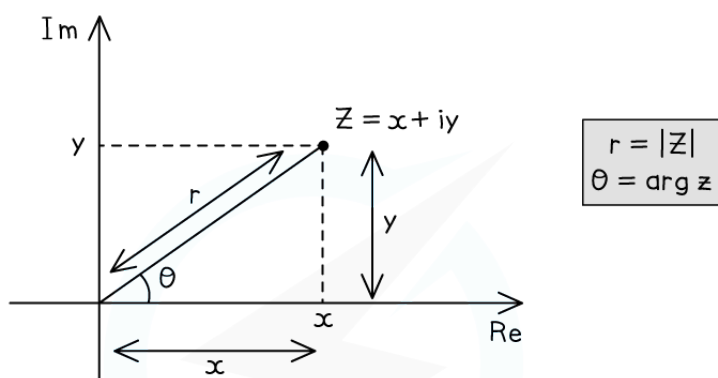
The complex number $Z = x + iy$ is said to be in Cartesian form. There are, however, other ways to write a complex number, such as in modulus-argument (polar) form.

How do I write a complex number in modulus-argument (polar) form?

- The **Cartesian form** of a complex number, $Z = x + iy$, is written in terms of its real part, x , and its imaginary part, y
- If we let $r = |z|$ and $\theta = \arg z$, then it is possible to write a complex number in terms of its modulus, r , and its argument, θ , called the **modulus-argument (polar) form**, given by...
 - $z = r(\cos \theta + i \sin \theta)$
- It is usual to give arguments in the range $-\pi < \theta \leq \pi$
 - Negative arguments should be shown clearly, e.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$ without simplifying $\cos\left(-\frac{\pi}{3}\right)$ to either $\cos\left(\frac{\pi}{3}\right)$ or $\frac{1}{2}$
 - Occasionally you could be asked to give arguments in the range $0 \leq \theta < 2\pi$
- If a complex number is given in the form $z = r(\cos \theta - i \sin \theta)$, then it is not currently in modulus-argument (polar) form due to the minus sign, but can be converted as follows...
 - By considering transformations of trigonometric functions, we see that $-\sin \theta \equiv \sin(-\theta)$ and $\cos \theta \equiv \cos(-\theta)$
 - Therefore $z = r(\cos \theta - i \sin \theta)$ can be written as $z = r(\cos(-\theta) + i \sin(-\theta))$, now in the correct form and indicating an argument of $-\theta$
- To convert from modulus-argument (polar) form back to Cartesian form, evaluate the real and imaginary parts
 - E.g. $z = 2\left(\cos\left(-\frac{\pi}{3}\right) + i \sin\left(-\frac{\pi}{3}\right)\right)$ becomes $z = 2\left(\frac{1}{2} + i\left(-\frac{\sqrt{3}}{2}\right)\right) = 1 - \sqrt{3}i$



Your notes



$$\cos \theta = \frac{x}{r} \Rightarrow x = r \cos \theta$$

$$\sin \theta = \frac{y}{r} \Rightarrow y = r \sin \theta$$

$$\therefore z = x + iy \Rightarrow z = r \cos \theta + ir \sin \theta$$

$$z = r(\cos \theta + i \sin \theta)$$

MODULUS – ARGUMENT (POLAR) FORM

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What are the rules for moduli and arguments under multiplication and division?

- When two complex numbers, z_1 and z_2 , are **multiplied** to give $z_1 z_2$, their **moduli** are also **multiplied**

$$|z_1 z_2| = |z_1| |z_2|$$

- When two complex numbers, z_1 and z_2 , are **divided** to give $\frac{z_1}{z_2}$, their **moduli** are also **divided**

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$$

- When two complex numbers, z_1 and z_2 , are **multiplied** to give $z_1 z_2$, their **arguments** are **added**

$$\arg(z_1 z_2) = \arg z_1 + \arg z_2$$

- When two complex numbers, z_1 and z_2 , are **divided** to give $\frac{z_1}{z_2}$, their **arguments** are **subtracted**

$$\arg\left(\frac{z_1}{z_2}\right) = \arg z_1 - \arg z_2$$



Your notes

How do I multiply complex numbers in modulus-argument (polar) form?

- The main benefit of writing complex numbers in modulus-argument (polar) form is that they multiply and divide very easily (often quicker than when in Cartesian form)
- To **multiply** two complex numbers, Z_1 and Z_2 , in modulus-argument (polar) form we use the rules from above to **multiply their moduli** and **add their arguments**
 - $|Z_1 Z_2| = |Z_1| |Z_2|$
 - $\arg(Z_1 Z_2) = \arg Z_1 + \arg Z_2$
- So if $Z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $Z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ then the rules above give...
 - $Z_1 Z_2 = r_1 r_2 (\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2))$
- Sometimes the new argument, $\theta_1 + \theta_2$, does not lie in the range $-\pi < \theta \leq \pi$ (or $0 \leq \theta < 2\pi$ if this is being used)
 - An out-of-range argument can be adjusted by either **adding or subtracting 2π**
 - E.g. If $\theta_1 = \frac{2\pi}{3}$ and $\theta_2 = \frac{\pi}{2}$ then $\theta_1 + \theta_2 = \frac{7\pi}{6}$
 - This is currently not in the range, but by subtracting 2π from $\frac{7\pi}{6}$ to give $-\frac{5\pi}{6}$, a new argument is formed that lies in the correct range and represents the same angle on an Argand diagram
- The rules of **multiplying the moduli** and **adding the arguments** can also be applied when...
 - ...multiplying three complex numbers together, $Z_1 Z_2 Z_3$, or more
 - ...finding powers of a complex number (e.g. Z^2 can be written as ZZ)
- Whilst not examinable, the rules for multiplication can be proved algebraically by multiplying $Z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ by $Z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, expanding the brackets and using compound angle formulae

How do I divide complex numbers in modulus-argument (polar) form?

- To **divide** two complex numbers, Z_1 and Z_2 in modulus-argument (polar) form, we use the rules from above to **divide their moduli** and **subtract their arguments**
 - $\left| \frac{Z_1}{Z_2} \right| = \frac{|Z_1|}{|Z_2|}$
 - $\arg\left(\frac{Z_1}{Z_2}\right) = \arg Z_1 - \arg Z_2$
- So if $Z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ and $Z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$ then the rules above give...

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} (\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2))$$

- As with multiplication, sometimes the new argument, $\theta_1 - \theta_2$, can lie out of the range $-\pi < \theta \leq \pi$ (or the range $0 \leq \theta < 2\pi$ if this is being used)
 - You can **add or subtract 2π** to bring out-of-range arguments back in range
- Whilst not examinable, the rules for division can be proved algebraically by dividing $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$ by $z_2 = r_2(\cos \theta_2 + i \sin \theta_2)$, using complex division and compound angle formulae



Your notes



Your notes

Worked example

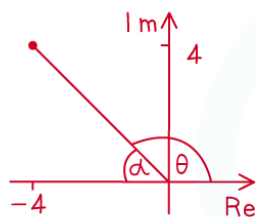


Let $z_1 = -4 + 4i$ and $z_2 = \sqrt{8} \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)$.

- Write z_1 in the form $r(\cos \theta + i \sin \theta)$, where r and θ are exact
- Find $z_1 z_2$, giving your answer in the form $r(\cos \theta + i \sin \theta)$ where $0 \leq \theta < 2\pi$
- Find $\frac{z_1}{z_2}$, giving your answer in the form $r(\cos \theta + i \sin \theta)$ where $-\pi < \theta \leq \pi$

a) $z_1 = -4 + 4i$;

SKETCH TO HELP

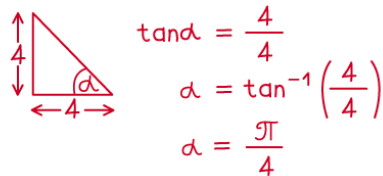


FIND $|z_1|$ BY PYTHAGORAS

$$|z_1| = \sqrt{(-4)^2 + 4^2} (= \sqrt{32})$$

$$= 4\sqrt{2} \quad] \text{ THIS IS } r$$

FIND ARG z_1 USING TRIGONOMETRY:



$$\arg z = \pi - \frac{\pi}{4} = \frac{3\pi}{4} \quad] \text{ THIS IS } \theta$$

WRITE IN MODULUS - ARGUMENT FORM:

$$z_1 = 4\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right)$$

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b) $z_1 = 4\sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right) \leftarrow \text{USING PART d)}$

$$z_2 = \sqrt{8} \left(\cos \left(-\frac{\pi}{2} \right) + i \sin \left(-\frac{\pi}{2} \right) \right)$$

MULTIPLY THE MODULI: $|z_1 z_2| = |z_1| |z_2|$

$$= 4\sqrt{2} \times \sqrt{8}$$

$$= 16$$

ADD THE ARGUMENTS: $\arg(z_1 z_2) = \arg z_1 + \arg z_2$

$$= \frac{3\pi}{4} + \left(-\frac{\pi}{2} \right)$$

Check range in question
 $0 \leq \theta < 2\pi$

$$= \frac{\pi}{4} \quad \checkmark \text{ IN RANGE}$$

$$z_1 z_2 = 16 \left(\cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right)$$

c) DIVIDE THE MODULI: $\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}$

$$= \frac{4\sqrt{2}}{\sqrt{8}}$$

$$= 2$$

SUBTRACT THE ARGUMENTS: $\arg \left(\frac{z_1}{z_2} \right) = \arg z_1 - \arg z_2$

$$= \frac{3\pi}{4} - \left(-\frac{\pi}{2} \right)$$

CHECK RANGE IN QUESTION
 $-\pi < \theta \leq \pi$

$$= \frac{5\pi}{4} \quad \times$$

NOT IN RANGE, SO
TRY SUBTRACTING 2π

$$\frac{5\pi}{4} - 2\pi = -\frac{3\pi}{4} \quad \checkmark \text{ IN RANGE}$$

$$\frac{z_1}{z_2} = 2 \left(\cos \left(-\frac{3\pi}{4} \right) + i \sin \left(-\frac{3\pi}{4} \right) \right)$$

 Examiner Tip

- The rules for multiplying and dividing in modulus-argument (polar) form must be learnt (they are not given in the formula booklet).
- Remember to add or subtract 2π to any out-of-range arguments to bring them back in range.
- If a question does not give a clear range for arguments, then both $-\pi < \theta \leq \pi$ or $0 < \theta \leq 2\pi$ would be accepted.



Your notes



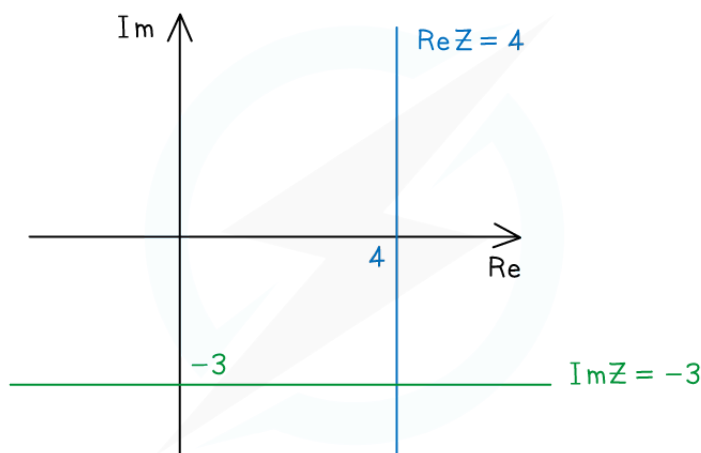
Your notes

8.2.4 Loci in Argand Diagrams

Loci in Argand Diagrams

How do I sketch the locus of $\text{Re } z = k$ or $\text{Im } z = k$ on an Argand diagram?

- All complex numbers, $z = x + iy$, that satisfy the equation $\text{Re } z = k$ lie on a **vertical line** with Cartesian equation $x = k$
 - Any complex number along this vertical line will have a **real part** of k
- All complex numbers, $z = x + iy$, that satisfy the equation $\text{Im } z = k$ lie on a **horizontal line** with Cartesian equation $y = k$
 - Any complex number along this horizontal line will have an **imaginary part** of k
- E.g. The loci $\text{Re } z = 4$ and $\text{Im } z = -3$ are represented by the vertical line $x = 4$ and the horizontal line $y = -3$



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Sketching the loci of $\text{Re } z = 4$ and $\text{Im } z = -3$

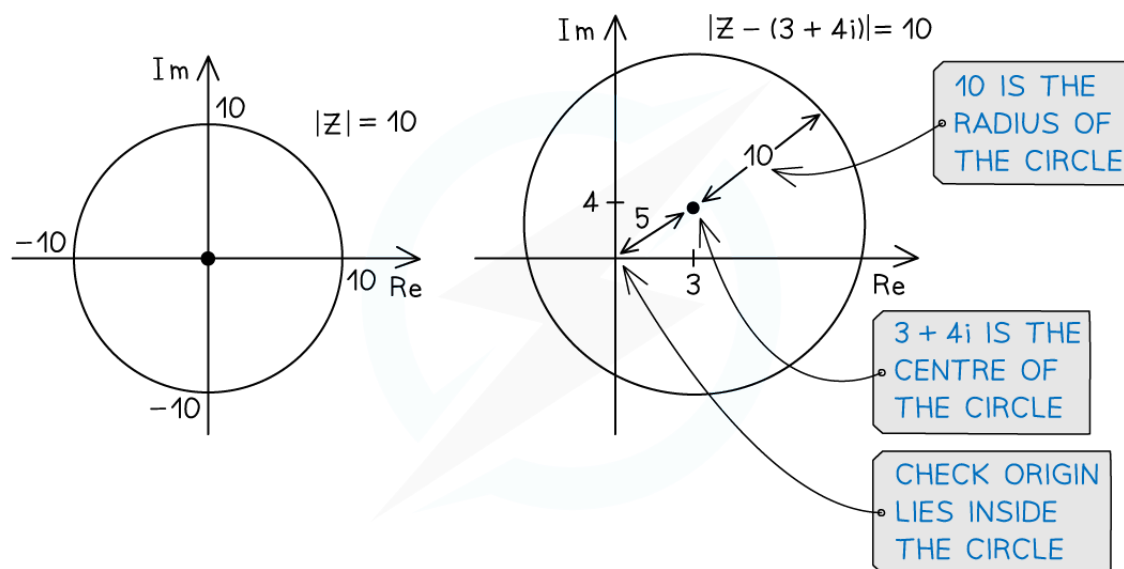
How do I sketch the locus of $|z - a| = k$ on an Argand diagram?

- All complex numbers, z , that satisfy the equation $|z| = k$ lie on a **circle** of **radius** k about the **origin**
 - E.g. the locus of $|z| = 10$ is a circle of radius 10, centred at the origin, as every complex number on that circle has a modulus of 10
- For a given complex number, a , all complex numbers, z , that satisfy the equation $|z - a| = k$ lie on a **circle** of **radius** k about the **centre** a
 - This is because $|z - a|$ represents the distance between complex numbers z and a



Your notes

- E.g. the locus of $|z - (3 + 4i)| = 10$ is a circle of radius 10 about $(3 + 4i)$
- Many equations need to be adjusted algebraically into the correct $|z - a|$ form
 - E.g. to find the centre of the circle $|z - 8i + 3| = 12$, first rewrite it as $|z - (-3 + 8i)| = 12$, giving the centre as $-3 + 8i$
 - E.g. to find the centre of the circle $|z + i| = 2$, first rewrite it as $|z - (-i)| = 2$, giving the centre as $(-i)$
 - Note that the centre of the circle $|z| = 5$ is the origin (it can be thought of as $|z - 0| = 5$)
- In order to sketch correctly, **check** whether the **origin** lies **outside**, **on** or **inside** the circle
 - E.g. for the locus of $|z - (3 + 4i)| = 10$, the distance from the centre of the circle, $3 + 4i$, to the origin is 5 (by Pythagoras), which is **less** than the radius of 10; a sketch must therefore show the origin **inside** the circle
- By knowing the radius and centre of a circle, the Cartesian equation of the circle can be found
 - The circle $|z - (3 + 4i)| = 10$ has a radius of 10 and centre of $(3, 4)$ in coordinates, so the equation of the circle is $(x - 3)^2 + (y - 4)^2 = 100$



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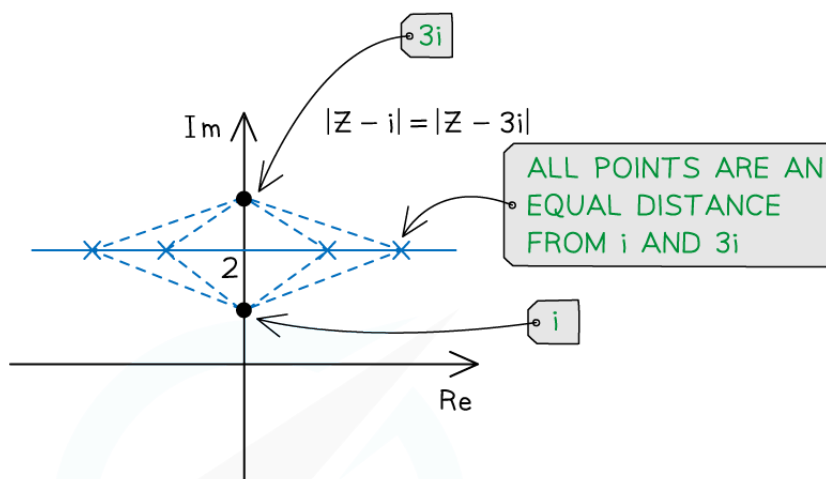
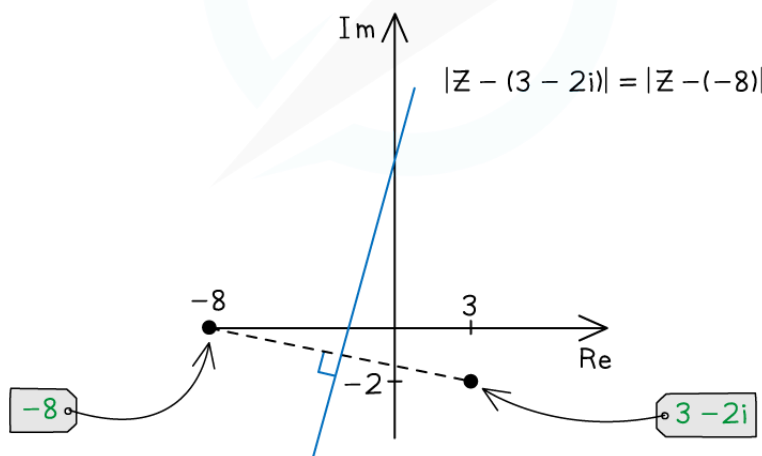
Sketching the loci of $|z| = 10$ and $|z - (3 + 4i)| = 10$

How do I sketch the locus of $|z - a| = |z - b|$ on an Argand diagram?

- For two given complex numbers, a and b , all complex numbers, z , that satisfy the equation $|z - a| = |z - b|$ lie on the **perpendicular bisector** of a and b
 - This is because the distance from z to a must equal the distance from z to b
 - a condition that is satisfied by all the complex numbers, z , on the perpendicular bisector of a and b

- E.g. the locus of $|z - 3 + 2i| = |z + 8|$ can be rewritten as $|z - (3 - 2i)| = |z - (-8)|$ which is the perpendicular bisector of the points $3 - 2i$ and -8
- A **sketch** of the perpendicular bisector is sufficient, without finding its exact equation (though this could be found using coordinate geometry methods)

PERPENDICULAR
BISECTOR OF
 i AND $3i$



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Sketching the loci of $|z - i| = |z - 3i|$ and $|z - (3 - 2i)| = |z - (-8)|$

How do I sketch the locus of $\arg(z - a) = \alpha$ on an Argand diagram?

- All complex numbers, Z , that satisfy the equation $\arg Z = \alpha$ lie on a **half-line** from the **origin** at an angle of α to the positive real axis

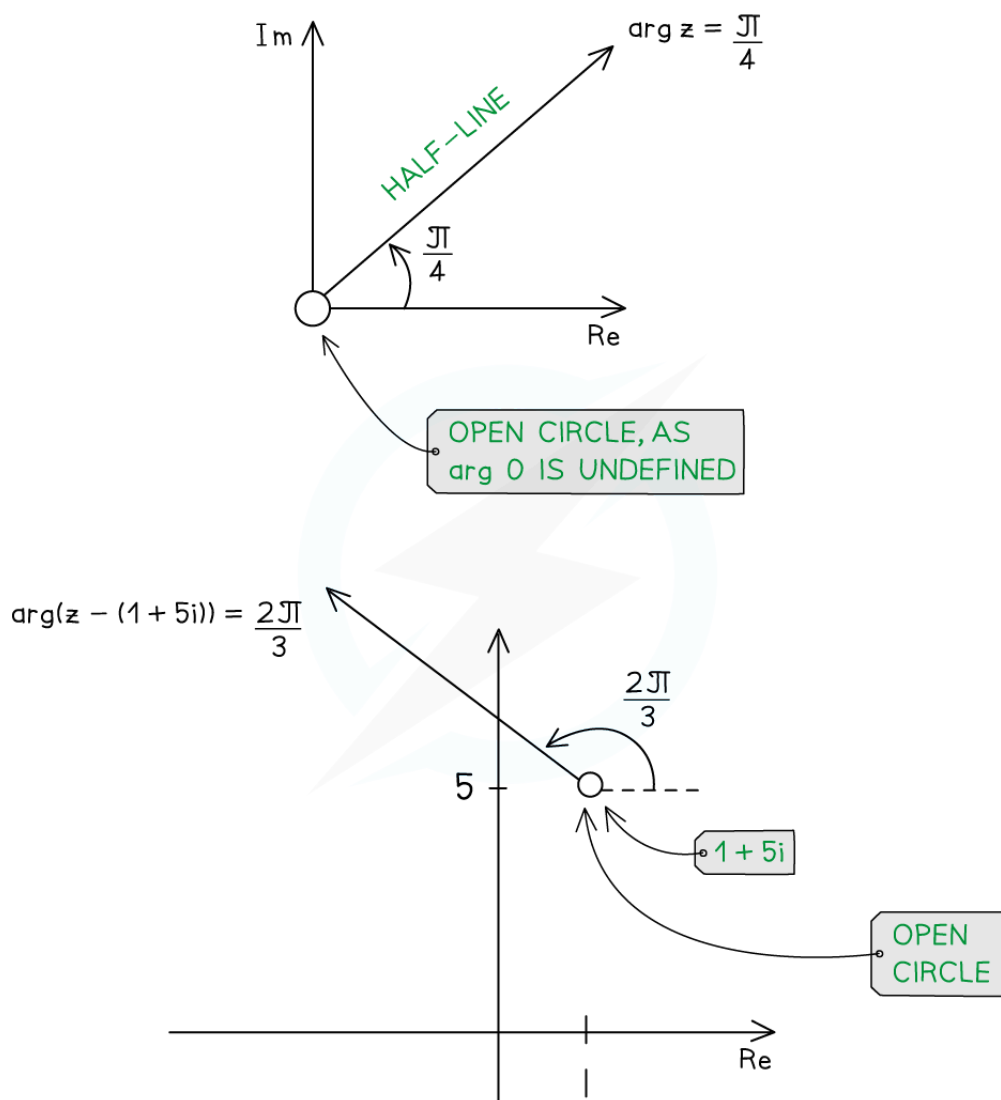


Your notes

- Although the half-line starts at the origin, the origin itself ($z = 0$) does not satisfy the equation $\arg z = \alpha$ as $\arg 0$ is undefined (there is no angle at the origin)
- To show the **exclusion** of $z = 0$ from the locus of $\arg z = \alpha$, a small **open circle** at the origin is used
- E.g. the locus of $\arg z = \frac{\pi}{4}$ is a half-line of angle $\frac{\pi}{4}$ to the positive real axis, starting from the origin, with an open circle at the origin
- For a given complex number, a , all complex numbers, z , that satisfy the equation $\arg(z - a) = \alpha$ lie on a **half-line** from the **point** a at an **angle** of α to the positive real axis, with an open circle to show the **exclusion** of $z = a$
 - E.g. the locus of $\arg(z - 1 - 5i) = \frac{2\pi}{3}$ can be rewritten as $\arg(z - (1 + 5i)) = \frac{2\pi}{3}$, which is a half-line of angle $\frac{2\pi}{3}$ measured from the point $1 + 5i$, with an open circle at $1 + 5i$ to show its exclusion
- In some cases, the **equation** of the half-line can be found using a **sketch** to help
 - E.g. the locus of $\arg z = \frac{\pi}{4}$ is the half-line $y = x$ for $x > 0$
 - E.g. the locus of $\arg(z - (8 + 5i)) = -\frac{\pi}{4}$ can be thought of, in coordinate geometry, as the half-line through $(8, 5)$ with gradient -1 , giving $y = -x + 13$ for $x > 8$
 - Whilst **not examinable**, the half-line equation for a more general angle, $\arg z = \alpha$, is $y = (\tan \alpha)x$ for $x > 0$, as the $\text{gradient} = \frac{\text{opposite}}{\text{adjacent}} = \tan \alpha$



Your notes



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Sketching the loci of $\arg z = \frac{\pi}{4}$ and $\arg(z - (1 + 5i)) = \frac{2\pi}{3}$

 Worked example



On separate axes, sketch the locus of points representing complex numbers, z , that satisfy the following equations:

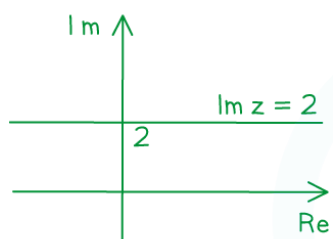
(a) $\text{Im } z = 2$

(b) $|z + 5 - 12i| = 8$

(c) $|z - 4i| = |z + i - 1|$

(d) $\arg(z + i) = -\frac{\pi}{3}$

a)



A HORIZONTAL LINE THROUGH 2 ON THE IMAGINARY AXIS

b)

$|z + 5 - 12i| = 8$

$|z - (-5 + 12i)| = 8$

NEED REARRANGING INTO THE FORM $|z - a|$

A CIRCLE OF RADIUS 8 WITH CENTRE $-5 + 12i$

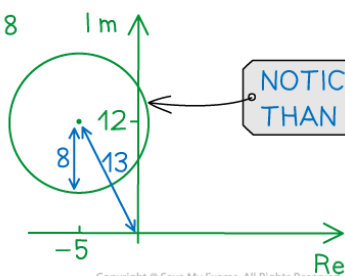
DISTANCE FROM CENTRE TO ORIGIN IS $\sqrt{(-5)^2 + 12^2} (= \sqrt{169})$

$= 13$

RADIUS IS $8 < 13 \therefore$ ORIGIN OUTSIDE CIRCLE

CHECK IF ORIGIN IS INSIDE, ON, OR OUTSIDE THE CIRCLE

$|z - (-5 + 12i)| = 8$



NOTICE RADIUS OF 8 IS GREATER THAN 5, SO CIRCLE OVERLAPS HERE

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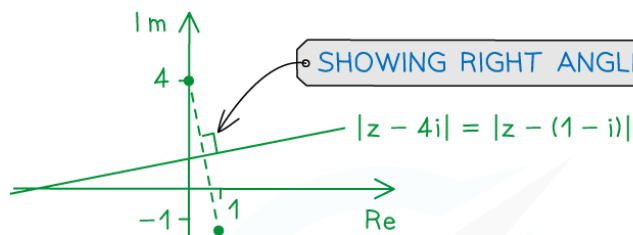


Your notes

c) $|z - 4i| = |z + i - 1|$
 $|z - 4i| = |z - (1 - i)|$

NEED REARRANGING INTO $|z - a| = |z - b|$

THE PERPENDICULAR BISECTOR OF $4i$ AND $1 - i$



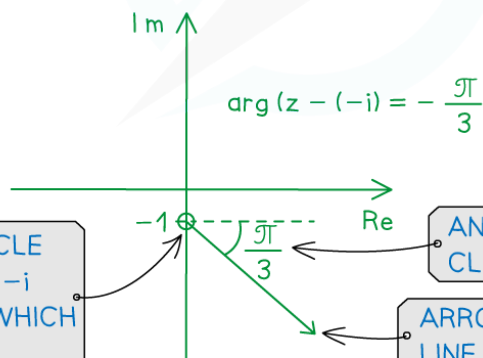
SHOWING RIGHT ANGLE ON SKETCH

d) $\arg(z + i) = -\frac{\pi}{3}$

NEEDS REARRANGING INTO THE FORM $\arg(z - a)$

$\arg(z - (-i)) = -\frac{\pi}{3}$

ANGLE IS $-\frac{\pi}{3}$



THE OPEN CIRCLE EXCLUDES $z = -i$ (GIVING $\arg 0$ WHICH IS UNDEFINED)

ANGLE SHOWN CLEARLY ON DIAGRAM

ARROWHEAD INDICATES HALF-LINE CONTINUES THIS WAY

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Examiner Tip

- In the exam, do not worry about making your diagrams perfect.
- A quick sketch with all the key features is sufficient.



Your notes

8.2.5 Inequalities & Regions in Argand Diagrams

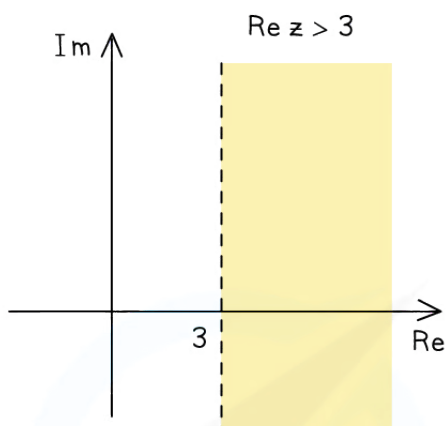
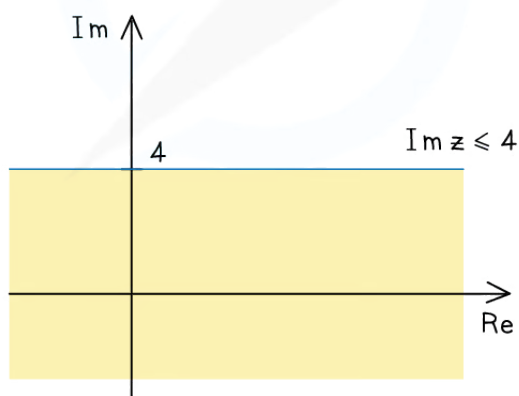
Inequalities & Regions in Argand Diagrams

How do I sketch inequalities such as $\text{Re } z < k$ or $\text{Im } z < k$ on an Argand diagram?

- The inequality $\text{Re } z < 2$ is satisfied by all complex numbers, $z = x + iy$, with a real part less than 2
 - On an Argand diagram, this would be the **region** to the left of the vertical line $x = 2$
 - The vertical line itself, $x = 2$, should be drawn dotted to show that points on this line are not permitted due to the strict inequality ($<$)
- In general, **dotted** lines are used with **strict** inequalities ($<$ or $>$) and **solid** lines are used with inequalities that **can be equal** ($\leq$ or $\geq$)
- Here is how to represent the following inequalities as regions on an Argand diagram...
 - $\text{Re } z < k$: **shade** the region to the **left** of the **dotted** vertical line (or **solid** vertical line for $\leq$)
 - $\text{Re } z > k$: **shade** the region to the **right** of the **dotted** vertical line (or **solid** vertical line for $\geq$)
- Similarly, here is how to represent the following inequalities as regions on an Argand diagram...
 - $\text{Im } z < k$: **shade** the region **below** the **dotted** horizontal line (or **solid** horizontal line for $\leq$)
 - $\text{Im } z > k$: **shade** the region **above** the **dotted** horizontal line (or **solid** horizontal line for $\geq$)



Your notes

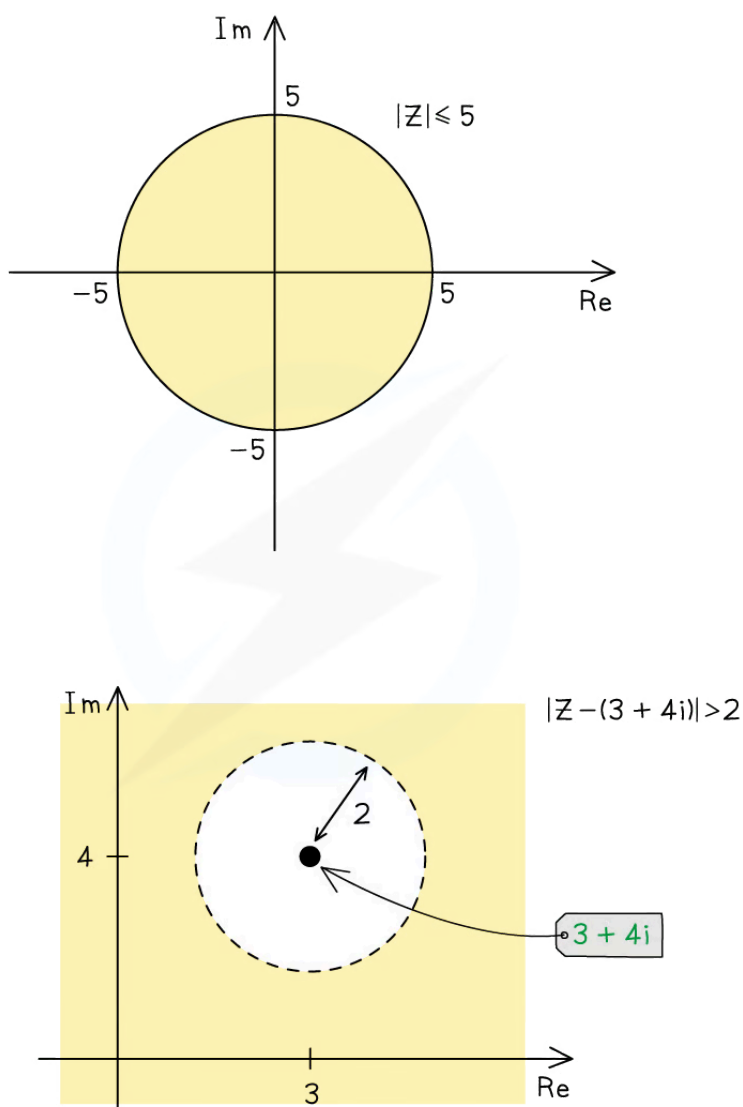


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Sketching the inequalities $\text{Re } z > 3$ and $\text{Im } z \leq 4$

How do I sketch inequalities such as $|z - a| < k$ on an Argand diagram?

- For a given complex number a , the inequality $|z - a| < k$ represents all complex numbers, z , that lie inside the circle of radius k , centred at a
- Here is how to represent the following inequalities as regions on an Argand diagram...
 - $|z - a| < k$: **shade** the region **inside** the circle of radius k , centred at a
 - $|z - a| > k$: **shade** the (unbounded) region **outside** circle of radius k , centred at a
 - Again use a **dotted** line for strict inequalities ($<$ and $>$) and a **solid** line for weak inequalities ($\leq$ or $\geq$)



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Sketching the inequalities $|z| \leq 5$ and $|z - (3 + 4i)| > 2$

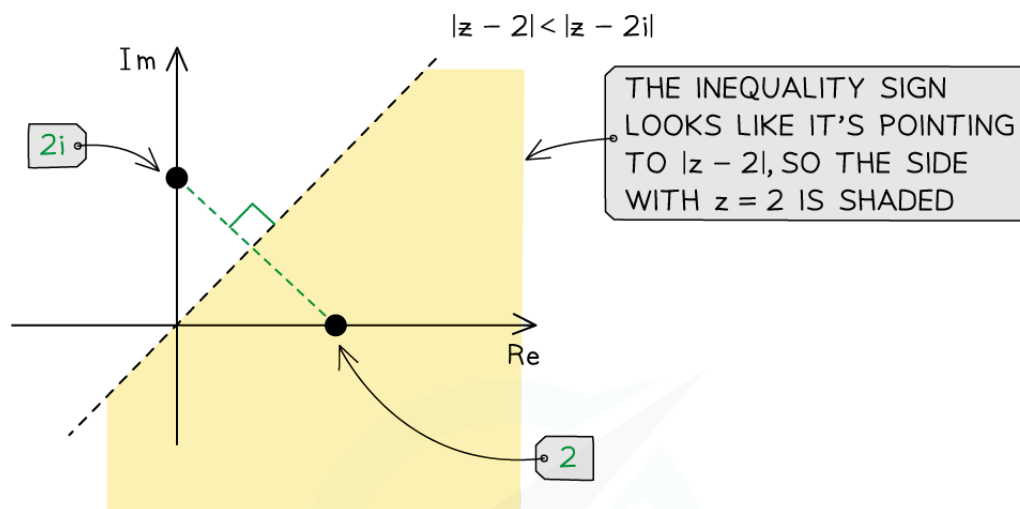
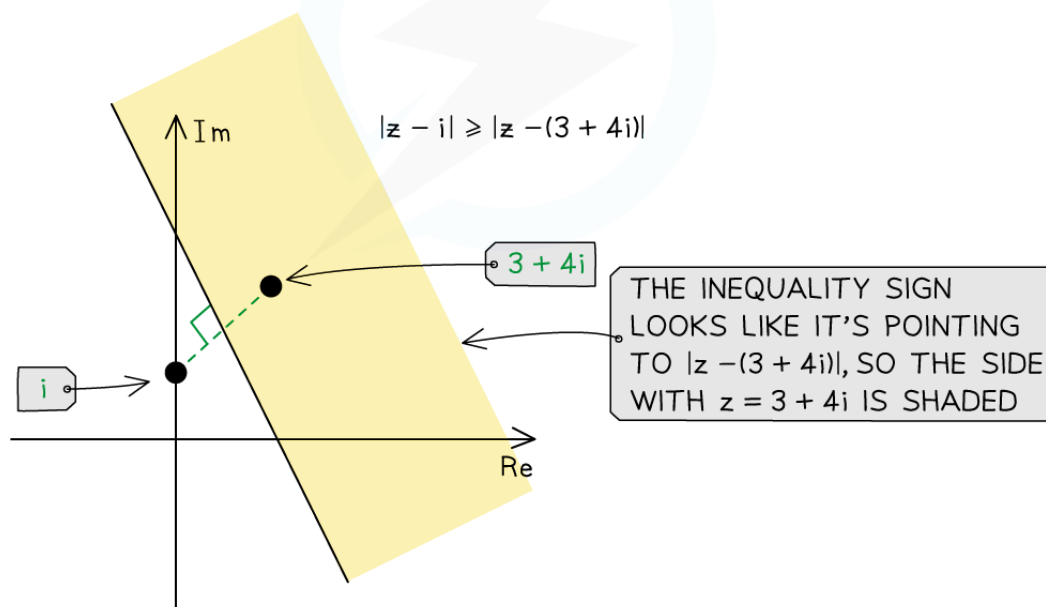
How do I sketch inequalities such as $|z - a| < |z - b|$ on an Argand diagram?

- For two given complex numbers a and b , the inequality $|z - a| < |z - b|$ represents all complex numbers, z , that lie closer to a than to b
- Here is how to represent the following inequalities as regions on an Argand diagram...
 - $|z - a| < |z - b|$: shade the region on the side of the perpendicular bisector of a and b that includes the point a



Your notes

- $|z - a| > |z - b|$: shade the region on the side of the dotted perpendicular bisector of a and b that includes the point b
- A good way to remember which side to shade is that the inequality sign points (like an arrow) to the side to be shaded



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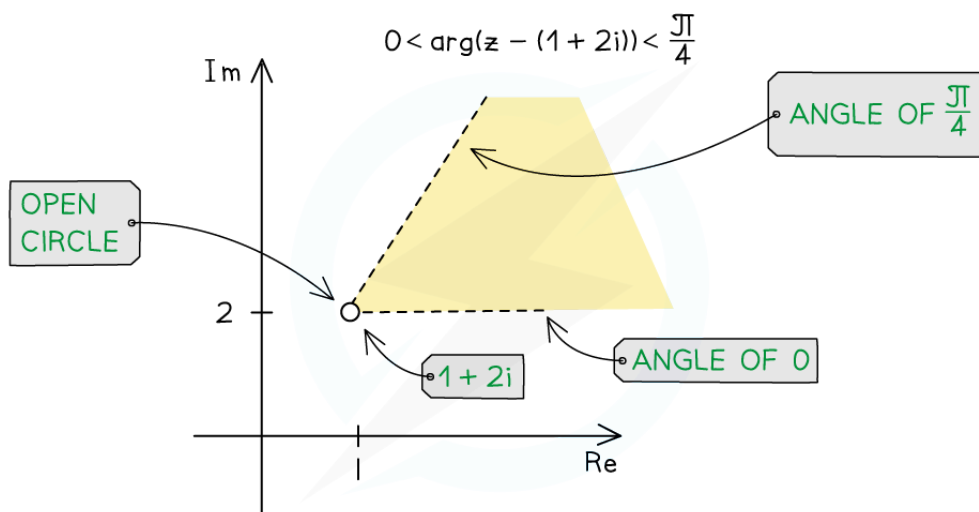
Sketching the inequalities $|z - 2| < |z - 2i|$ and $|z - i| \geq |z - (3 + 4i)|$

How do I sketch inequalities such as $\alpha < \arg(z - a) < \beta$ on an Argand diagram?



Your notes

- For a given complex number, a , the inequality $\alpha < \arg(z - a) < \beta$ represents all complex numbers, z , that have an **argument** between α and β , as **measured from** the point a
- Here is how to represent the following inequalities as regions on an Argand diagram...
 - $\alpha < \arg(z - a) < \beta$: shade the **wedge-shaped region** between the half-line $\arg(z - a) = \alpha$ and the half-line $\arg(z - a) = \beta$, with an open circle at $z = a$ to show the **exclusion** of this point (to avoid the undefined value of $\arg 0$)



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Sketching the inequalities $0 < \arg(z - (1 + 2i)) < \frac{\pi}{4}$

How do I draw multiple inequalities on the same Argand diagram?

- To sketch a region that satisfies multiple inequalities, we need to find the **intersection** of all the regions (where all regions overlap)
 - E.g. the region satisfied by $2 < |z| \leq 5$ can be found as follows...
 - Using diagonal lines in the same direction, lightly shade the region $|z| > 2$ (the outside of a circle of radius 2 about the origin)
 - Using diagonal lines in a different direction, lightly shade the region $|z| \leq 5$ (the inside of a circle of radius 5 about the origin)
 - Where the diagonal lines **cross each other** highlights the region satisfying both inequalities; this should be shaded **boldly**
 - E.g. the region satisfied by $|z - i| < |z - 1|$ and $0 \leq \arg z \leq \frac{3\pi}{4}$ can be found as follows...
 - Using diagonal lines in the same direction, lightly shade the region $|z - i| < |z - 1|$ (the side including $z = i$ from the perpendicular bisector of the points i and 1)

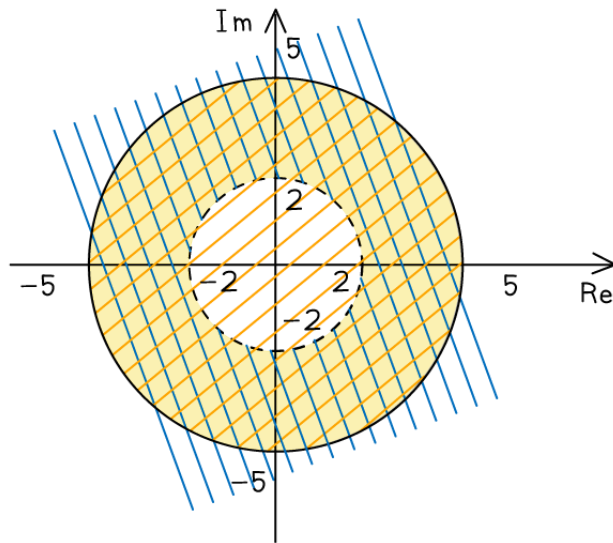
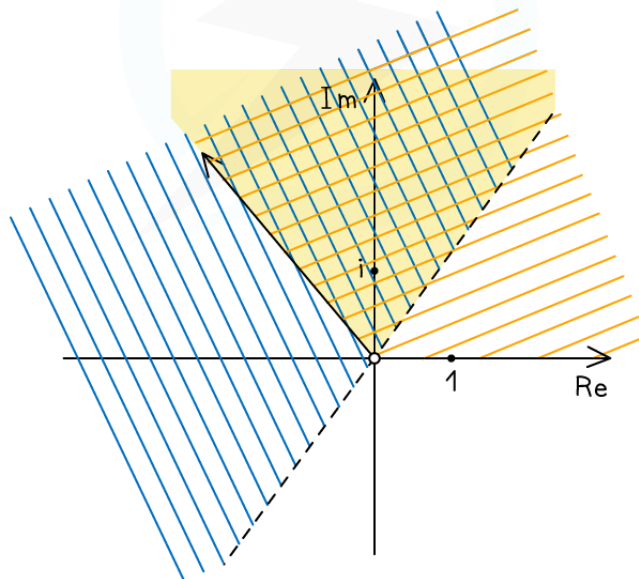
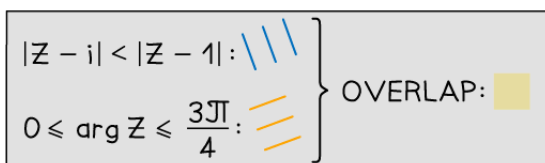


Your notes

- Using diagonal lines in a different direction, lightly shade the region $0 \leq \arg z \leq \frac{3\pi}{4}$ (the wedge-shaped region from 0 to $\frac{3\pi}{4}$ radians from the origin)
- Where the diagonal lines **cross each other** highlights the region satisfying both inequalities; this should be shaded **boldly**



Your notes



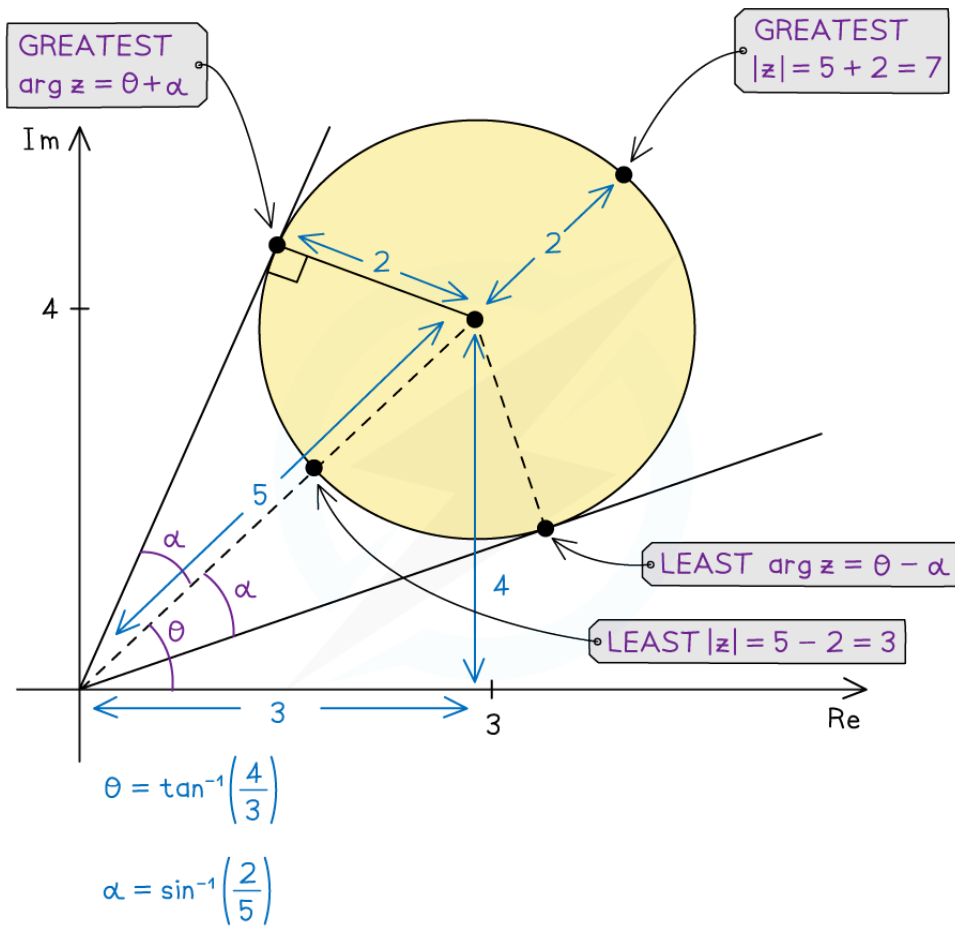


Your notes

The inequalities $2 < |z| \leq 5$ on the left, the inequalities $|z - i| < |z - 1|$ and $0 \leq \arg z < \frac{3\pi}{4}$ on the right

How do I find the greatest (or least) values of $|z|$ or $\arg z$ in a region?

- Every complex number, z , that lies within a region on an Argand diagram will have its own modulus, $|z|$, that is measured from the origin
- For any shaped region...
 - The **least** value of $|z|$ is the **distance** from the **origin** to the **nearest point** in the region
 - The **greatest** value of $|z|$ is the **distance** from the **origin** to the **farthest point** in the region
 - Sometimes the least value is zero (if the origin is in the region) and/or the greatest value is infinite (if the region is unbounded)
- For circular regions that do not contain the origin...
 - Find the length from the origin to the centre of the circle...
 - then add a radius for the greatest value of $|z|$ or subtract a radius for the least value of $|z|$
- Every complex number, z , that lies within a region on an Argand diagram will have its own argument, $\arg z$, as measured from the origin
- For any shaped region...
 - The **least** value of $\arg z$ is the **smallest angle** a line through the origin can make to a point in the region
 - The **greatest** value of $\arg z$ is the **largest angle** a line through the origin can make to a point in the region
- For circular regions that do not contain the origin...
 - Find θ , the argument of the centre...
 - Find α , the angle between a tangent to the circle through the origin and the line from the origin to the centre...
 - This can be done using trigonometry, as a radius meets a tangent at right angles
 - The greatest value of $\arg z$ is $\theta + \alpha$ and the least value of $\arg z$ is $\theta - \alpha$



Find greatest and least values of $|z|$ and $\arg z$



Your notes

Worked example



(a) On separate Argand diagrams, shade the region whose points represent complex numbers z satisfying...

i. ...the inequality $|z - 4 - 4i| < |z|$

ii. ...both inequalities $|z + 1 - i| < 1$ and $-\frac{\pi}{4} \leq \arg(z + 2 - i) \leq \frac{\pi}{4}$

(b) For complex numbers z satisfying the inequality $|z - 5i| \leq 2$, find...

i. ...the greatest value of $|z|$

ii. ...the least value of $\arg z$

a) (i) $|z - 4 - 4i| < |z|$

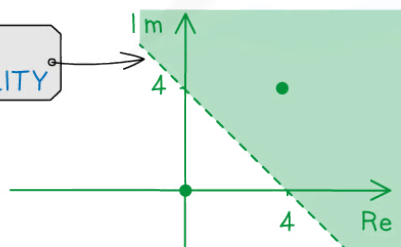
REARRANGE INTO THE FORM $|z - a| < |z - b|$

THIS IS LIKE $|z - 0|$

$|z - (4 + 4i)| < |z|$

NEED PERPENDICULAR BISECTOR OF $4 + 4i$ AND 0

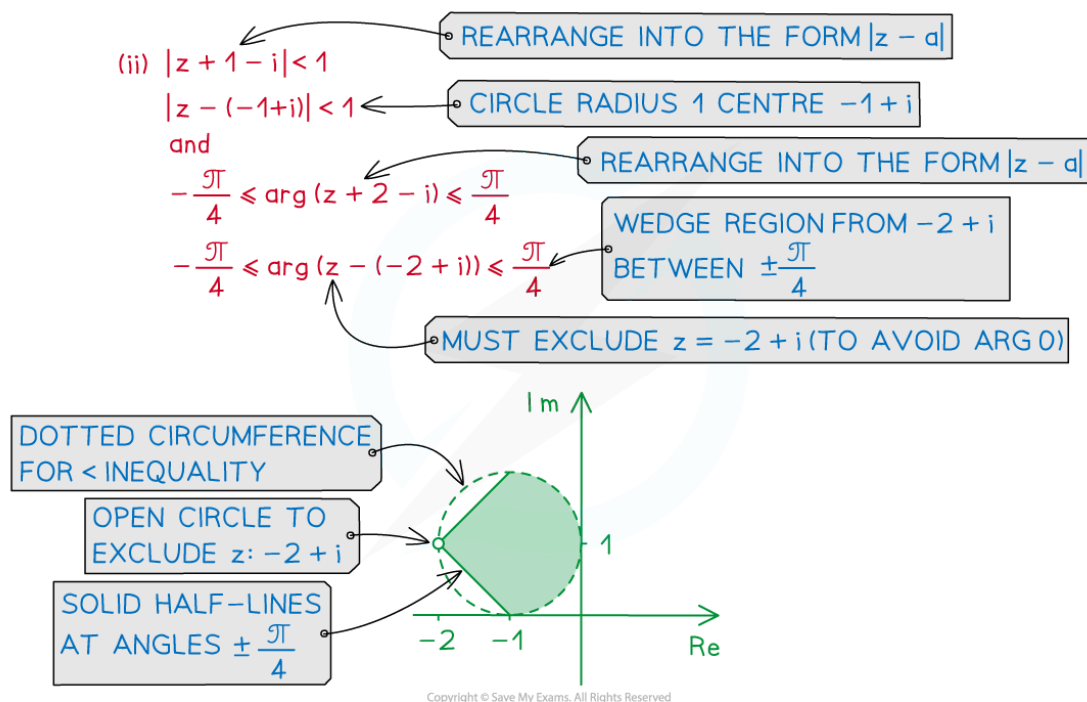
DOTTED LINE FOR $<$ INEQUALITY



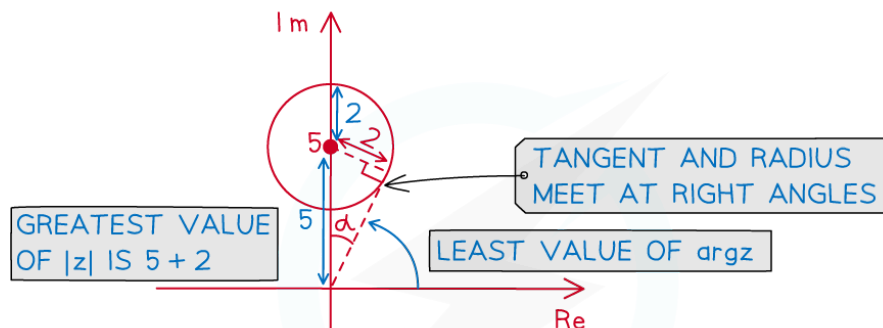
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Your notes



- b) $|z - 5i| \leq 2$
CIRCLE RADIUS 2 CENTRE $5i$
SKETCH TO HELP



(i) GREATEST VALUE OF $|z|$ IS 7

(ii) $\sin \alpha = \frac{2}{5}$ USING TRIGONOMETRY

$$\alpha = \sin^{-1} \left(\frac{2}{5} \right)$$

$$= 0.4115\dots$$

$$\frac{\pi}{2} - 0.4115\dots = 1.159\dots$$

MEASURING THE ARGUMENT
FROM THE POSITIVE REAL AXIS

LEAST VALUE OF $\text{ARG } z$ IS 1.16 RADIANS TO 3sf

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Examiner Tip

- When shading an unbounded region in the exam, make sure you extend your shading outwards, crossing any axes where necessary.
- When shading multiple inequalities to find a common region, lots of methods are accepted; these include lightly shading individual regions to see where they overlap, or starting to shade along the boundaries of individual regions to find the common region, or marking individual regions to find the region with multiple marks, etc.