CIE IGCSE Additional Maths

Your notes

Integration

Contents

- Introduction to Integration
- Integrating Powers of x
- Definite Integrals
- Finding Areas with Integration
- Finding Areas Between Lines & Curves
- Integrating Trig Functions
- Integrating e^x & 1/x
- Reverse Chain Rule

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Introduction to Integration

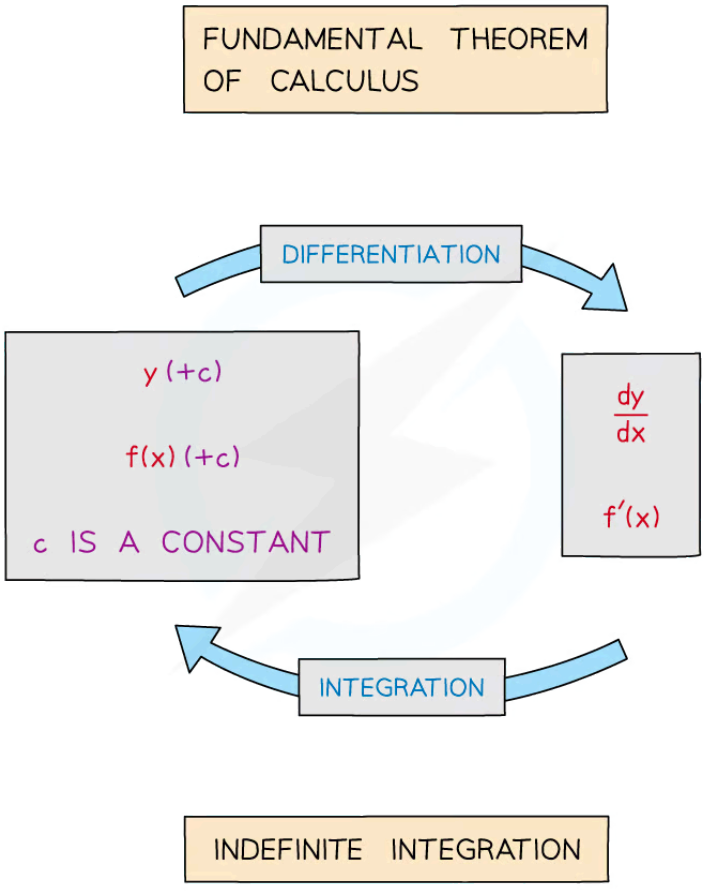
Your notes

Introduction to Integration

What is integration?

- **Integration** is the opposite to **differentiation**
- **Integration** is the process of finding the expression of a function from an expression of the **derivative** (**gradient function**)

What is the fundamental theorem of calculus?



Copyright © Save My Exams. All Rights Reserved

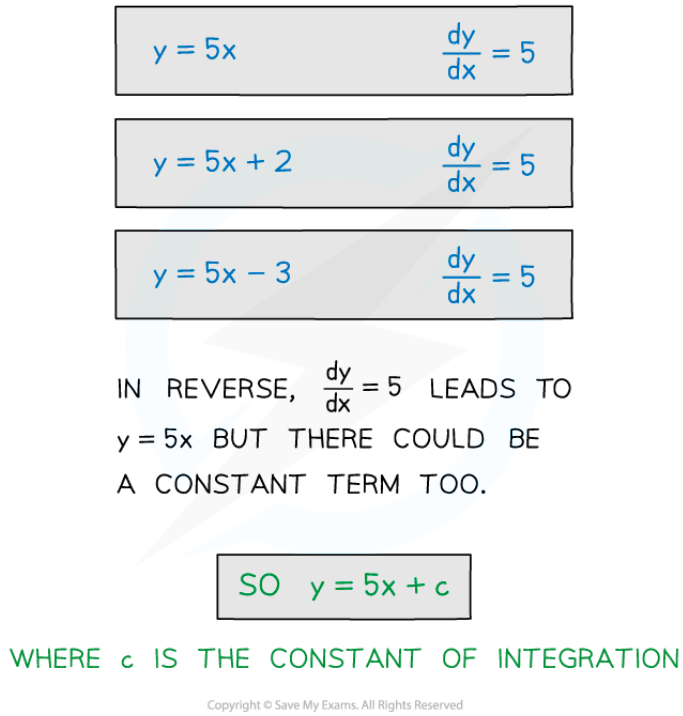
- The Fundamental Theorem of Calculus states that integration is the inverse process of differentiation
  - This form of the Theorem relates to **Indefinite Integration**
  - An alternative version of the Fundamental Theorem of Calculus involves **Definite Integration**

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

What is the constant of integration (+c)?

Your notes

- When differentiating ***y***, constant terms 'disappear'
  - for constants ***y* = c**, $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$
  - graphs of constants are horizontal lines and so have gradient of 0
- Integrating $\frac{\mathrm{d}y}{\mathrm{d}x}$, to get ***y***, cannot determine the constant
  - To acknowledge this constant, "**+ c**" is used
  - **c** is called the **constant of integration**



Copyright © Save My Exams. All Rights Reserved

What is the notation for integration?

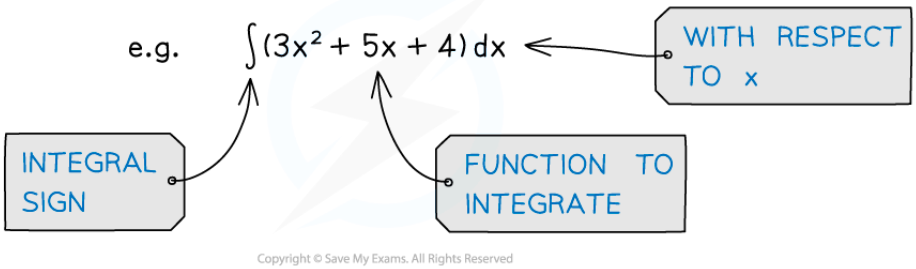
- An **integral** is normally written in the form

$$\int f(x)\,\mathrm{d}x$$

  - the large operator $\int$ means "integrate"
  - "$\mathrm{d}x$" indicates which variable to integrate with respect to
  - $f(x)$ is the function to be integrated
- If it has more than one term the function to be integrated (called the **integrand**) should be in brackets

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- "Integrate" -- "**all** of (...)" -- "with respect to x"



Copyright © Save My Exams. All Rights Reserved

Your notes

Worked example

(a) Find $\frac{dy}{dx}$ for $y = 6x + 3$

(b) Find $\int 6\,dx$

a) $\frac{dy}{dx} = 6$

b) $y = 6x + c$

CONSTANT OF INTEGRATION

Copyright © Save My Exams. All Rights Reserved

save my exams

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

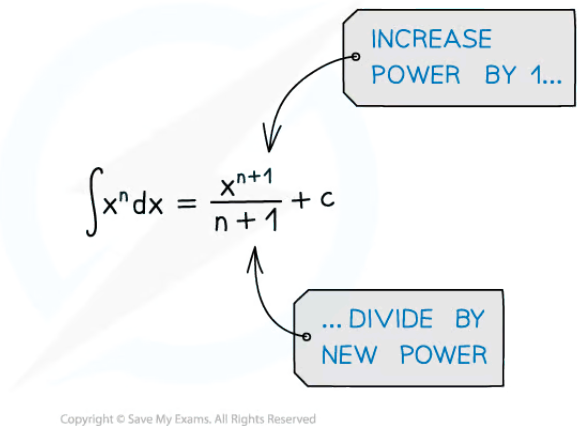
Integrating Powers of x

Your notes

Integrating Powers of x

How do I integrate powers of x?

- Powers of x are integrated according to the following formulae:
  - If $f(x) = x^n$ then $\int f(x)\,\mathrm{d}x = \frac{x^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and c is the **constant of integration**
  - For each term ...
    - ... increase the power (of x) by 1
    - ... divide by the new power
  - This does not apply when the original power is -1
    - the new power would be 0 and division by 0 is undefined



Copyright © Save My Exams. All Rights Reserved

- If the power of x is multiplied by a constant then the integral is also multiplied by that constant
  - If $f(x) = ax^n$ then $\int f(x)\,\mathrm{d}x = \frac{ax^{n+1}}{n+1} + c$ where $n \in \mathbb{Q}$, $n \neq -1$ and a is a constant and c is the **constant** of **integration**
- Remember the special case:
  - $\int a\,\mathrm{d}x = ax + c$
    - e.g. $\int 4\,\mathrm{d}x = 4x + c$
  - This allows **constant** terms to be integrated

How do I integrate expressions containing powers of x?

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- The formulae for integrating powers of x apply to **all rational numbers** so it is possible to integrate any expression that is a sum or difference of powers of x
  - e.g. If $f(x) = 8x^3 - 2x + 4$ then
    $$\int f(x)\,\mathrm{d}x = \frac{8x^{3+1}}{3+1} - \frac{2x^{1+1}}{1+1} + 4x + c = 2x^4 - x^2 + 4x + c$$
- Functions involving **roots** will need to be rewritten as **fractional powers** of x first
  - eg. If $f(x) = 5\sqrt[3]{x}$ then rewrite as $f(x) = 5x^{\frac{1}{3}}$ and integrate
- Functions involving **fractions** with **denominators** in **terms** of x will need to be rewritten as **negative powers** of x first
  - e.g. If $f(x) = \frac{4}{x^2} + x^2$ then rewrite as $f(x) = 4x^{-2} + x^2$ and integrate
- **Products** and **quotients** cannot be integrated this way so would need **expanding/simplifying** first
  - e.g. If $f(x) = 8x^2(2x - 3)$ then
    $$\int f(x)\,\mathrm{d}x = \int (16x^3 - 24x^2)\,\mathrm{d}x = \frac{16x^4}{4} - \frac{24x^3}{3} + c = 4x^4 - 8x^3 + c$$

Examiner Tip

- You can speed up the process of integration in the exam by committing the pattern of basic integration to memory
  - In general you can think of it as 'raising the power by one and dividing by the new power'
  - Practice this lots before your exam so that it comes quickly and naturally when doing more complicated integration questions

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

Given that

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - \frac{1}{\sqrt{x}}$$

find an expression for y in terms of x.

Rewrite all terms as powers of x using the laws of indices for fractional and negative powers on the last term.

$$\frac{dy}{dx} = 3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}$$

Find y by integrating each term.

$$y = \int \left(3x^4 - 2x^2 + 3 - x^{-\frac{1}{2}}\right) dx$$

$$y = \frac{3x^5}{5} - \frac{2x^3}{3} + 3x - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Rewrite using the same format given in the question.

$$y = \frac{3}{5}x^5 - \frac{2}{3}x^3 + 3x - 2\sqrt{x} + c$$

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Finding the Constant of Integration

How do I find the constant of integration?

Your notes

- **STEP 1**
  - Rewrite the function into a more easily integrable form
    - Each term needs to be a power of **x** (or a constant)
- **STEP 2**
  - Integrate each term and remember "+c"
    - Increase power by 1 and divide by new power
- **STEP 3**
  - Substitute the coordinates of a given point in to form an equation in **c**
    - Solve the equation to find **c**

e.g. FIND THE EQUATION OF THE GRAPH THAT PASSES THROUGH (2,6) AND HAS GRADIENT FUNCTION $3x^2(2x-1)$

ANOTHER NAME FOR $\frac{dy}{dx}$

STEP 1 REWRITE IN A MORE EASILY INTEGRABLE FORM

$$\frac{dy}{dx} = 3x^2(2x-1)$$

$$= 6x^3 - 3x^2$$

STEP 2 INTEGRATE...

$$y = \frac{6x^4}{4} - \frac{3x^3}{3} + c$$

... REMEMBERING "+c"

$$y = \frac{3x^4}{2} - x^3 + c$$

Copyright © Save My Exams. All Rights Reserved

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

STEP 3 USE THE GIVEN POINT TO FIND c

AT (2,6) $6 = \frac{3}{2}(2)^4 - 2^3 + c$

$c = -10$

$\therefore \quad y = \frac{3}{2}x^4 - x^3 - 10$

Copyright © Save My Exams. All Rights Reserved

Examiner Tip

- If a constant of integration can be found then the question will need to give you some extra information
  - If this is given then make sure you use it to find the value of c

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

Given $f'(x) = \frac{(x+3)^2}{\sqrt{x}}$ and $f(1) = 25$, find $f(x)$.

Rewrite $f'(x)$ in a form the can be integrated more easily.

$$f'(x) = \frac{(x+3)(x+3)}{\sqrt{x}}$$

$$= \frac{x^2 + 6x + 9}{x^{\frac{1}{2}}}$$

$$= \frac{x^2}{x^{\frac{1}{2}}} + \frac{6x}{x^{\frac{1}{2}}} + \frac{9}{x^{\frac{1}{2}}}$$

$$= x^{\frac{3}{2}} + 6x^{\frac{1}{2}} + 9x^{-\frac{1}{2}}$$

Integrate each term, remember to include a constant of integration.

$$f(x) = \frac{x^{\frac{5}{2}}}{\frac{5}{2}} + \frac{6x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{9x^{\frac{1}{2}}}{\frac{1}{2}} + c$$

Simplify.

$$f(x) = \frac{2}{5}x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + c$$

Use f(1) =25 to find the value of c.

$$f(1) = \frac{2}{5}(1)^{\frac{5}{2}} + 4(1)^{\frac{3}{2}} + 18(1)^{\frac{1}{2}} + c$$

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

$$\frac{2}{5} + 4 + 18 + c = 25$$

$$22\frac{2}{5} + c = 25$$

$$c = \frac{13}{5}$$

$$f(x) = \frac{2}{5}x^{\frac{5}{2}} + 4x^{\frac{3}{2}} + 18x^{\frac{1}{2}} + \frac{13}{5}$$

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

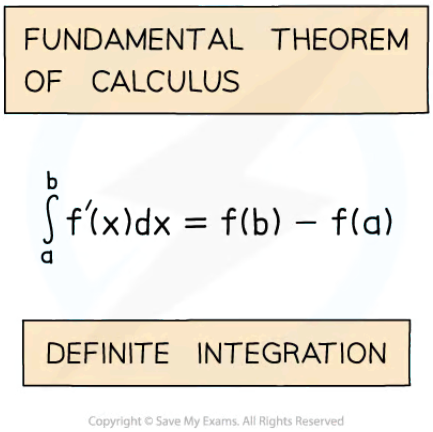
Definite Integrals

Your notes

Definite Integration

What is definite integration?

- **Definite Integration** occurs in an alternative version of the Fundamental Theorem of Calculus
- This version of the Theorem is the one referred to by most textbooks/websites



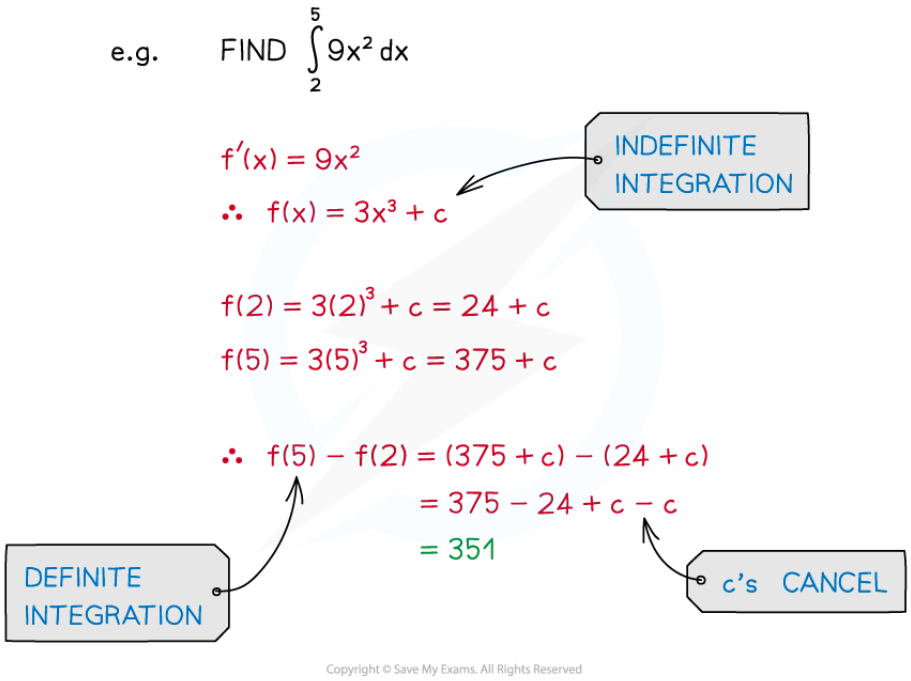
Copyright © Save My Exams. All Rights Reserved

- **a** and **b** are called limits
  - **a** is the lower limit
  - **b** is the upper limit
- **f'(x)** is the **derivative** of **f(x)**
- The value can be **positive, zero** or **negative**

Why do I not need to include a constant of integration for definite integration?

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



- "**+c**" would appear in both **f(a)** and **f(b)**
  - Since we then calculate **f(b)** – **f(a)** they cancel each other out
  - So "**+c**" is not included with definite integration

How do I find a definite integral?

- **STEP 1**
  - Give the integral a name (if it does not already have one)
    - This saves you having to rewrite the whole integral every time!
- **STEP 2**
  - If necessary rewrite the integral into a more easily integrable form
    - Not all functions can be integrated directly
- **STEP 3**
  - Integrate without applying the limits
    - Notation: use square brackets [ ] with limits placed after the end bracket
- **STEP 4**
  - Substitute the limits into the function and calculate the answer
    - Substitute the top limit first
    - Then substitute the bottom limit
    - Subtract the second value from the first

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

e.g. EVALUATE $\int_1^9 (2x^2 + \frac{3}{x^2} - 4\sqrt{x})\,dx$

STEP 1 NAME THE INTEGRAL, I

LET $I = \int_1^9 (2x^2 + \frac{3}{x^2} - 4\sqrt{x})\,dx$

STEP 2 REWRITE IN A MORE EASILY INTEGRABLE FORM

$I = \int_1^9 (2x^2 + 3x^{-2} - 4x^{\frac{1}{2}})\,dx$

STEP 3 INTEGRATE

- USE SQUARE BRACKETS
- LIMITS AT END

$I = \left[\frac{2}{3}x^3 - 3x^{-1} - \frac{8}{3}x^{\frac{3}{2}}\right]_1^9$

STEP 4 SUBSTITUTE LIMITS AND CALCULATE

$I = \left(486 - \frac{1}{3} - 72\right) - \left(\frac{2}{3} - 3 - \frac{8}{3}\right)$

$I = \frac{1256}{3}$

Copyright © Save My Exams. All Rights Reserved

What are the special properties of definite integrals?

- Some of these have been encountered already and some may seem obvious ...
  - taking **constant** factors outside the integral
    - $\int_a^b kf(x)\,dx = k\int_a^b f(x)\,dx$ where k is a constant
    - useful when fractional and/or negative values involved
  - integrating term by term

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- $\int_a^b [f(x) + g(x)]\,dx = \int_a^b f(x)\,dx + \int_a^b g(x)\,dx$
  - the above works for subtraction of terms/functions too
- equal upper and lower limits
  - $\int_a^a f(x)\,dx = 0$
  - on evaluating, this would be a value subtracted from itself!
- swapping limits gives the same, but negative, result
  - $\int_a^b f(x)\,dx = -\int_b^a f(x)\,dx$
  - compare 8 subtract 5 say, with 5 subtract 8 ...
- splitting the interval
  - $\int_a^b f(x)\,dx = \int_a^c f(x)\,dx + \int_c^b f(x)\,dx$ where $a \leq c \leq b$
  - this is particularly useful for areas under multiple curves or areas under the x-axis


Your notes

Examiner Tip

- Look out for questions that ask you to find an **indefinite** integral in one part (so "**+c**" needed), then in a later part use the same integral as a **definite** integral (where "**+c**" is not needed)

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

Your notes

Find the exact value of $\int_0^8 (3x^{\frac{1}{2}} + \frac{4}{x^3} - \frac{\sqrt{x}}{x^2})\,dx$

STEPS 1 & 2

$$I = \int_0^8 \left(3x^{\frac{1}{2}} + 4x^{-3} - x^{-\frac{3}{2}}\right)dx$$

- CALL INTEGRAL I
- REWRITE

STEP 3

INTEGRATE

$$I = \left[\frac{3x^{\frac{3}{2}}}{\frac{3}{2}} + \frac{4x^{-2}}{-2} - \frac{x^{-\frac{1}{2}}}{-\frac{1}{2}}\right]_0^8$$

$$I = \left[2x^{\frac{3}{2}} - 2x^{-2} + 2x^{-\frac{1}{2}}\right]_0^8$$

STEP 4

SUBSTITUTE LIMITS AND CALCULATE

$$I = \left[2(8)^{\frac{3}{2}} - 2(8)^{-2} + 2(8)^{-\frac{1}{2}}\right]$$
$$- \left[2(0)^{\frac{3}{2}} - 2(0)^{-2} + 2(0)^{-\frac{1}{2}}\right]$$

$$I = \left(32\sqrt{2} - \frac{1}{32} + \frac{\sqrt{2}}{2}\right) - (0-0+0)$$

$$I = \frac{-1 + 1040\sqrt{2}}{32}$$

Copyright © Save My Exams. All Rights Reserved
save my exams

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

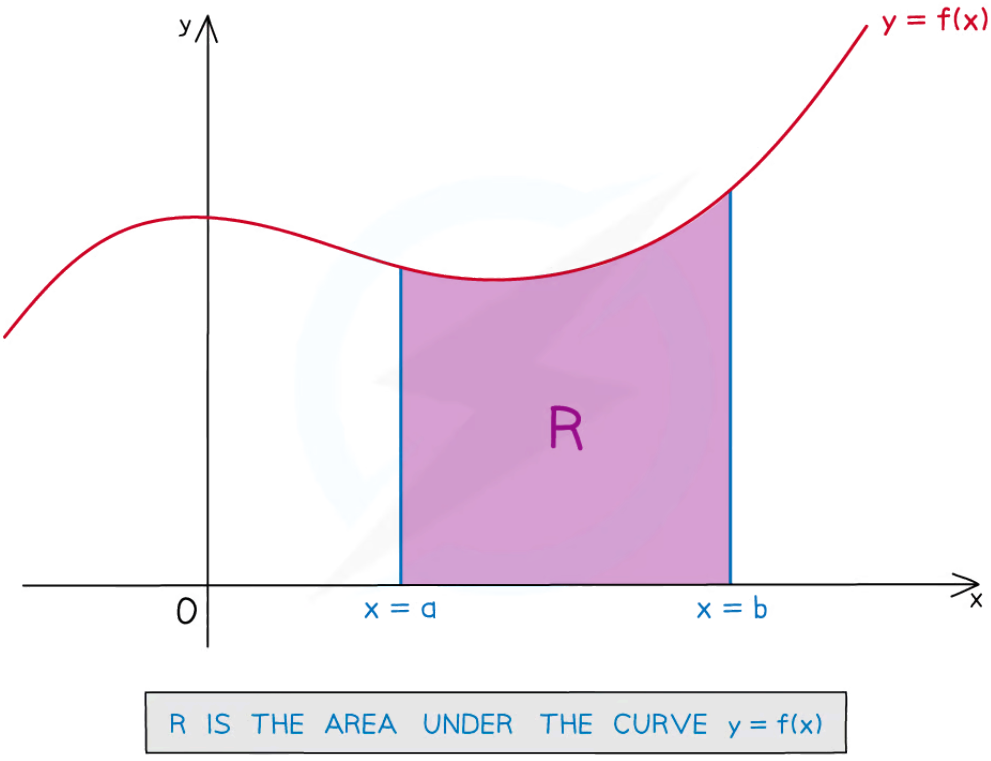
Finding Areas with Integration

Your notes

Area Under a Curve

What does area under a curve mean?

- The phrase "area under a curve" refers to the area bounded by ...
  - ... the **x**-axis
  - ... the graph of **y = f(x)**
  - ... the vertical line **x = a**
  - ... the vertical line **x = b**

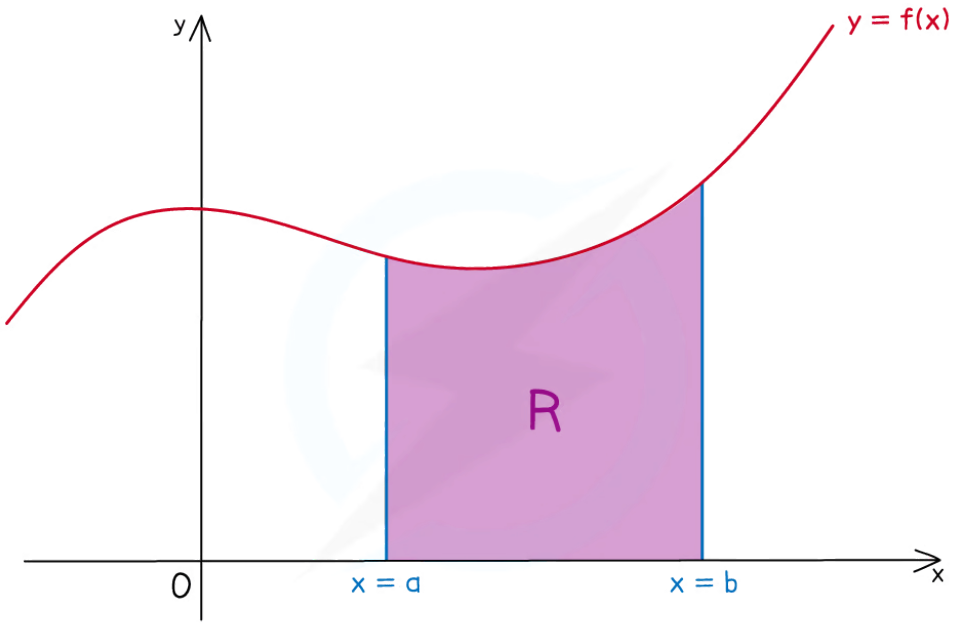


Copyright © Save My Exams. All Rights Reserved

How do I find the area under a curve?

- The value from **definite integration** is equal to the "area under a curve"

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers



$$\text{AREA OF } R = \int_a^b f(x)\,dx$$

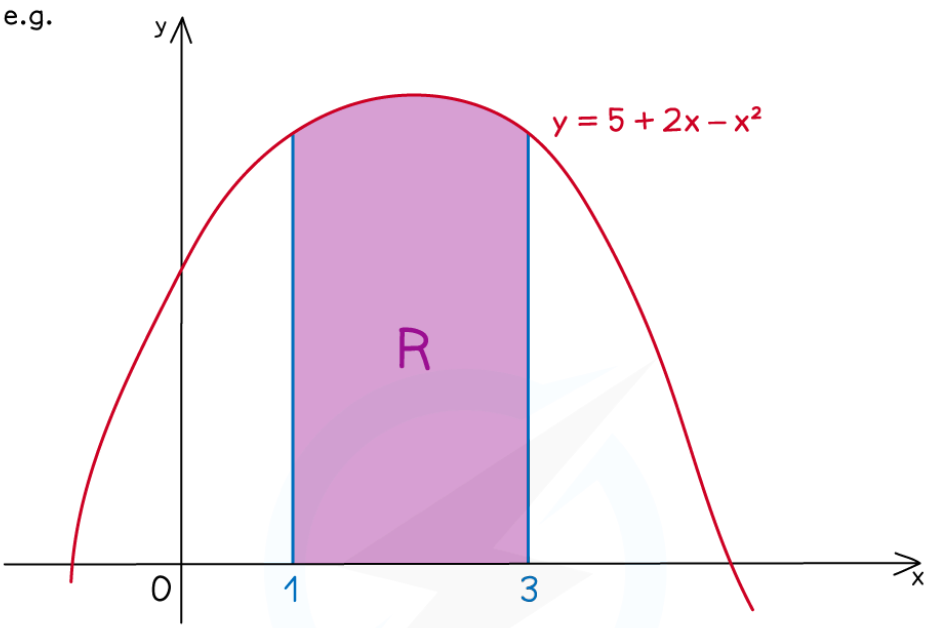
Copyright © Save My Exams. All Rights Reserved

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

e.g.



FIND THE AREA OF THE SHADED REGION.

$$\text{AREA} = \int_1^3 (5 + 2x - x^2)dx$$

$$= \left[5x + x^2 - \frac{1}{3}x^3\right]_1^3$$

$$= (15 + 9 - 9) - (5 + 1 - \frac{1}{3})$$

$$\text{AREA} = \frac{28}{3} \text{ SQUARE UNITS}$$

Copyright © Save My Exams. All Rights Reserved

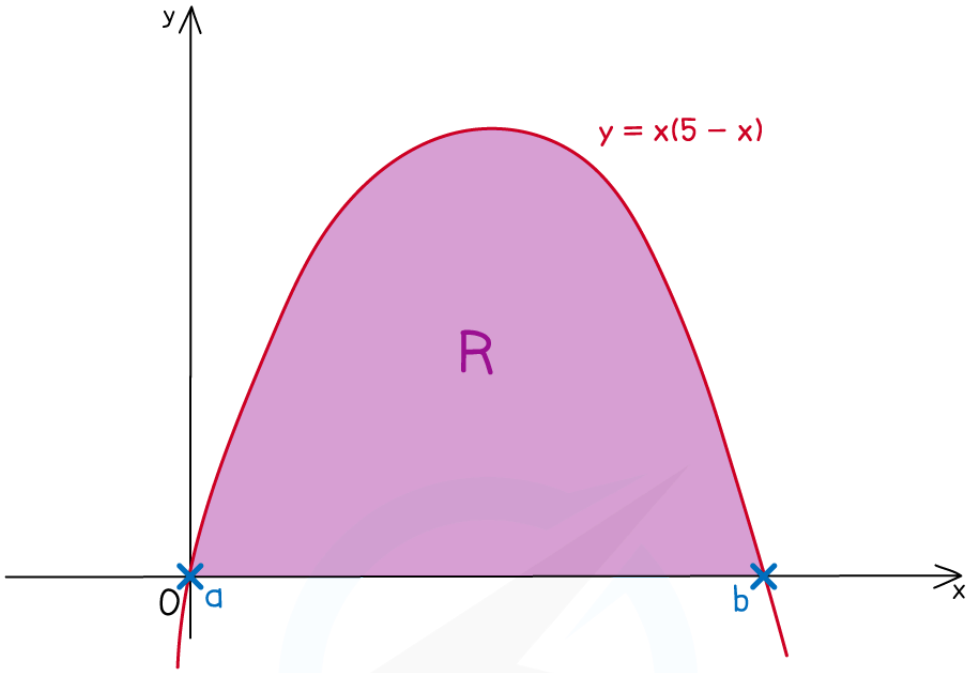
What if I am not told the limits?

- If limits are not provided they will likely be the **x**-axis intercepts
  - Set **y = 0** and solve the equation to find the **x**-axis intercepts

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

e.g. FIND THE SHADED AREA MARKED R IN THE DIAGRAM BELOW



$x(5-x)=0$ ← SET $y=0$ FOR x-AXIS INTERCEPTS

$x=0 \quad x=5$

$\therefore \quad a=0 \quad b=5$

EXPAND SO "INTEGRATE-ABLE"

$$\text{AREA OF R} = \int_0^5 (5x-x^2)\,dx$$

$$= \left[\frac{5x^2}{2} - \frac{x^3}{3}\right]_0^5$$

$$= \frac{125}{2} - \frac{125}{3} - (0-0)$$

$$\text{AREA OF R} = \frac{125}{6} \text{ SQUARE UNITS}$$

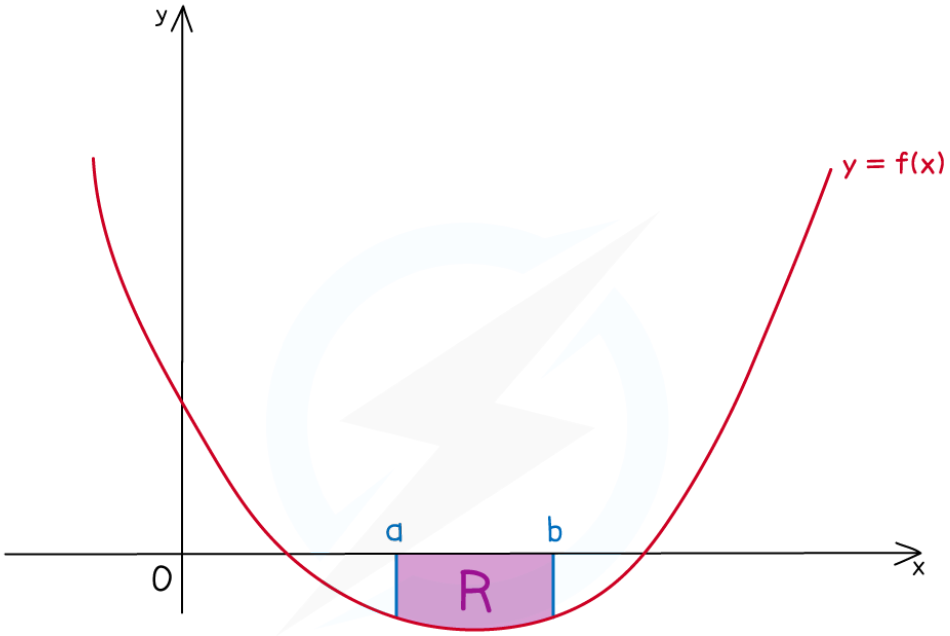
Copyright © Save My Exams. All Rights Reserved

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

What happens if the graph is below the x-axis?

- If the area lies underneath the **x**-axis the value of the integral will be negative
- An **area** cannot be negative, so take the **modulus** of the integral

Your notes



AREA R ENTIRELY UNDER x-AXIS

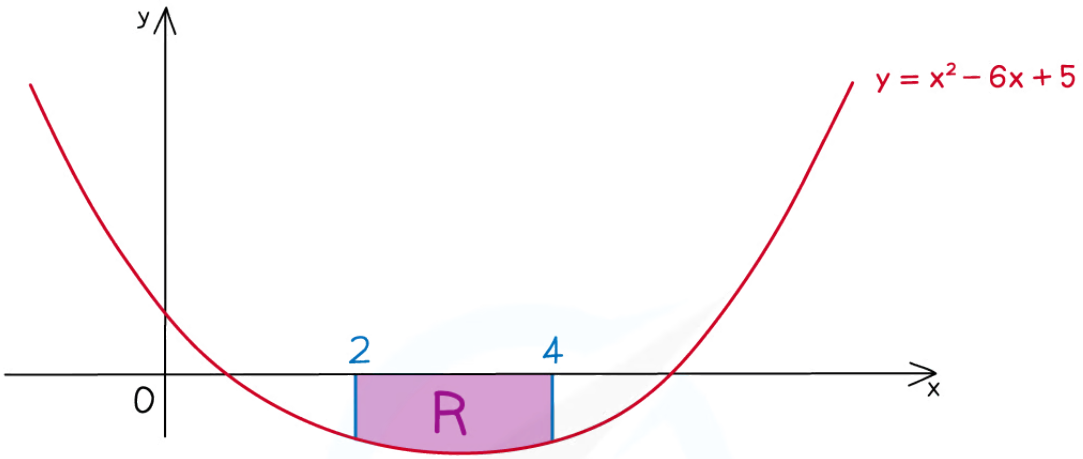
$$\int_a^b f(x)dx < 0$$

Copyright © Save My Exams. All Rights Reserved

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

e.g. FIND THE SHADED AREA R IN THE DIAGRAM BELOW



$$\int_2^4 (x^2 - 6x + 5)\,dx = \left[\frac{x^3}{3} - 3x^2 + 5x\right]_2^4$$

$$= \left(\frac{64}{3} - 48 + 20\right) - \left(\frac{8}{3} - 12 + 10\right)$$

$$= -\frac{22}{3}$$

WRITE AS AN INTEGRAL

$\therefore$ AREA OF R $= \frac{22}{3}$ SQUARE UNITS

$\left|-\frac{22}{3}\right| = \frac{22}{3}$

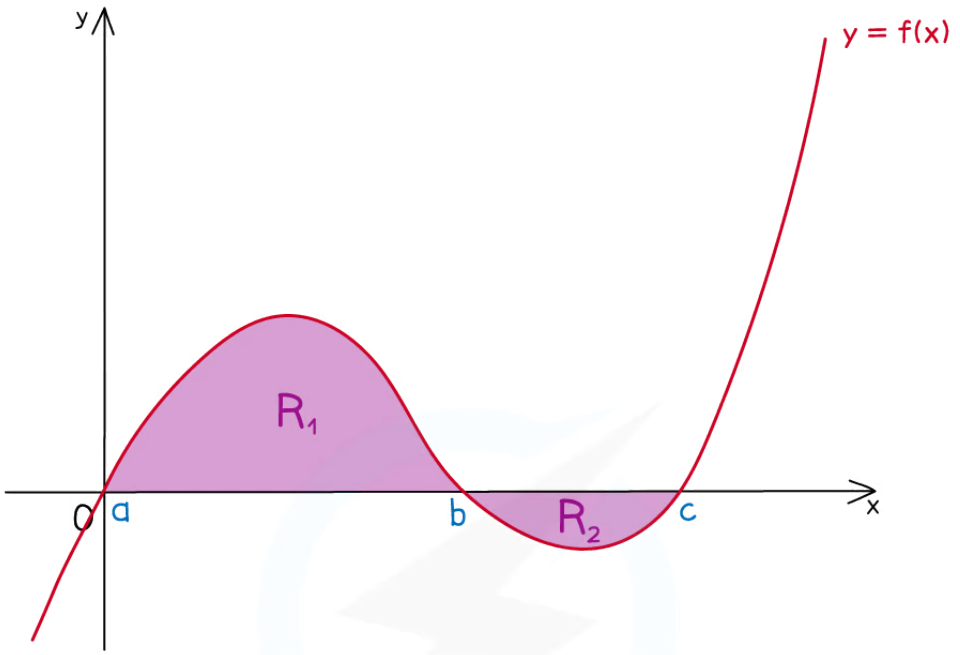
Copyright © Save My Exams. All Rights Reserved

What if the area is made up of more than one section?

- Be careful when one section is above and one section is below the **x**-axis
  - You will need a separate integral for each section, BUT...
  - ...One section's integral will be **negative**
  - ...One section's integral will be **positive**
  - So you'll need to take the **modulus** before adding to find the total area

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



TOTAL AREA $= R_1 + R_2$

AND $\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$

BUT $\int_a^b f(x)dx > 0$

$\int_b^c f(x)dx < 0$

SO TOTAL AREA $\neq \int_a^c f(x)dx$

Copyright © Save My Exams. All Rights Reserved

Examiner Tip

- Add information to any diagram provided in the question, as well as axes intercepts and values of limits
- Mark and shade the area you're trying to find, and if no diagram is provided, **sketch** one!

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

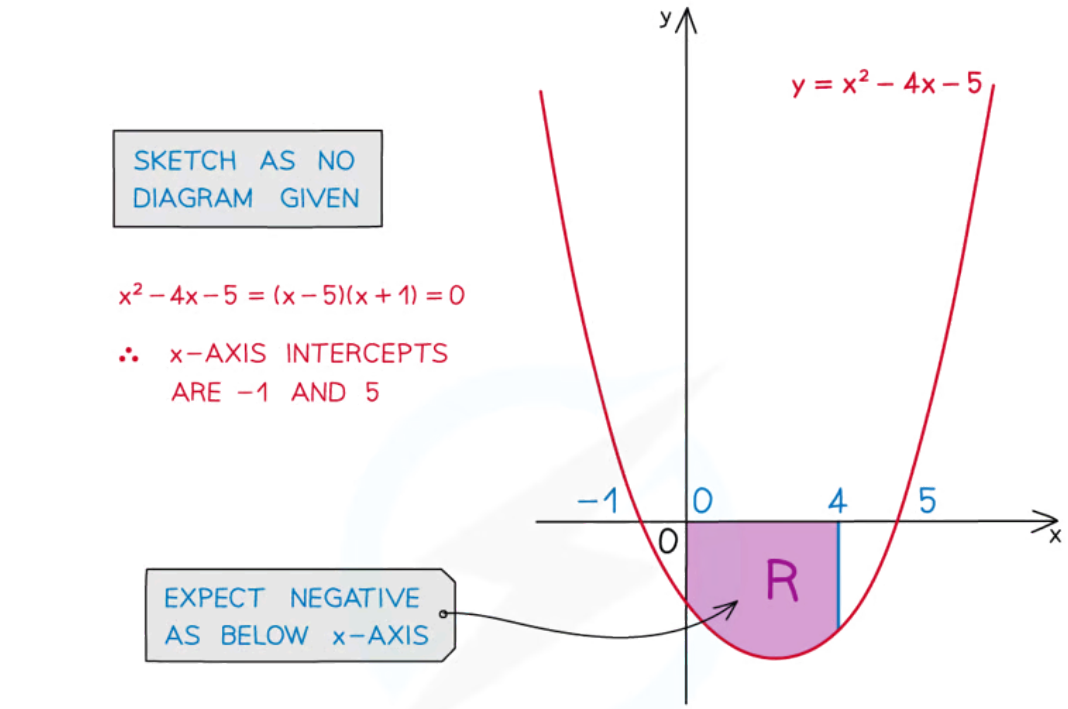
Worked example

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Find the area bounded by the curve with equation $y = x^2 - 4x - 5$, the x-axis and the lines $x = 0$ and $x = 4$.



SKETCH AS NO DIAGRAM GIVEN

$x^2 - 4x - 5 = (x - 5)(x + 1) = 0$

$\therefore$ x−AXIS INTERCEPTS ARE −1 AND 5

EXPECT NEGATIVE AS BELOW x−AXIS

$$\int_0^4 (x^2 - 4x - 5)\,dx = \left[\frac{x^3}{3} - 2x^2 - 5x\right]_0^4$$

WRITE AS AN INTEGRAL

$$= \left(\frac{64}{3} - 32 - 20\right) - (0 - 0 - 0)$$

$$= -\frac{92}{3}$$

$\therefore$ AREA REQUIRED $= \frac{92}{3}$ SQUARE UNITS

MODULUS

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Copyright © Save My Exams. All Rights Reserved
save my exams

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

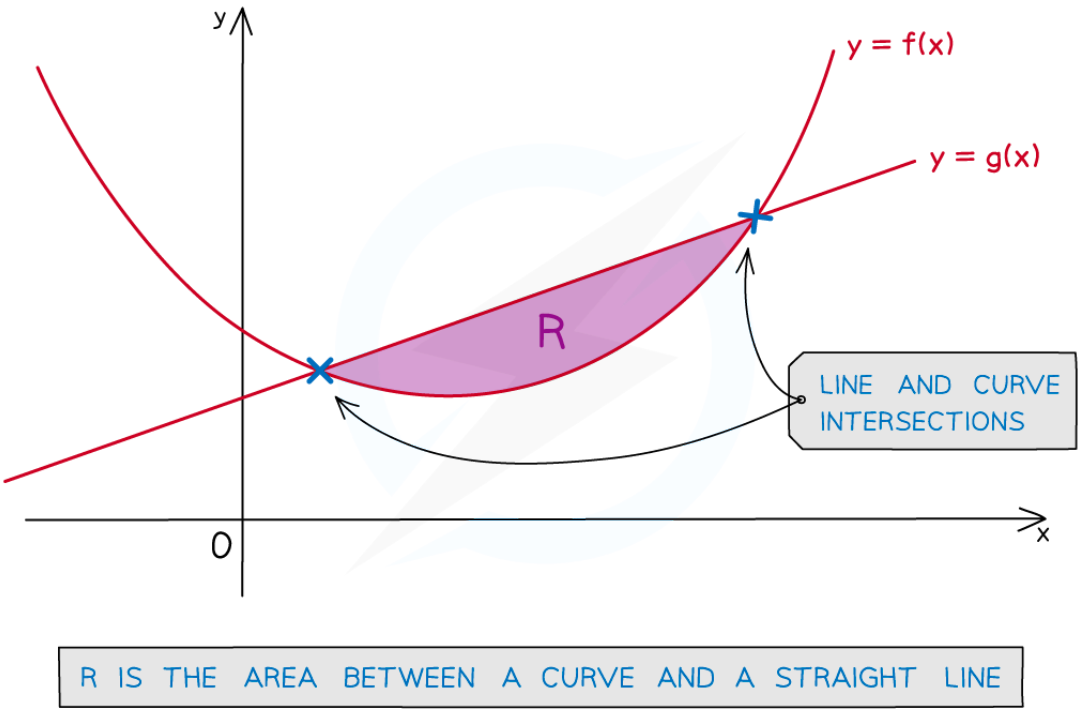
Finding Areas Between Lines & Curves

Your notes

Area Between a Curve & a Line

How do I find the area between a curve and a line?

- The area enclosed will be the **difference** between ...
  - the **area under** the **line** and
  - the **area under** the **curve**

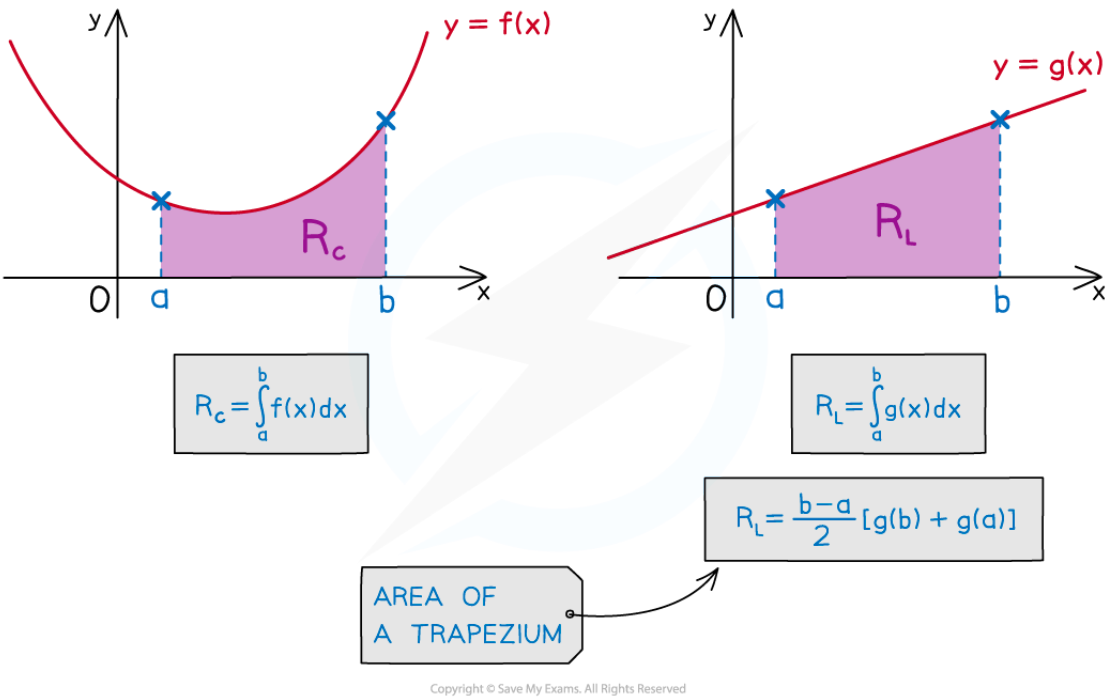


Copyright © Save My Exams. All Rights Reserved

- These can be found separately

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



Copyright © Save My Exams. All Rights Reserved

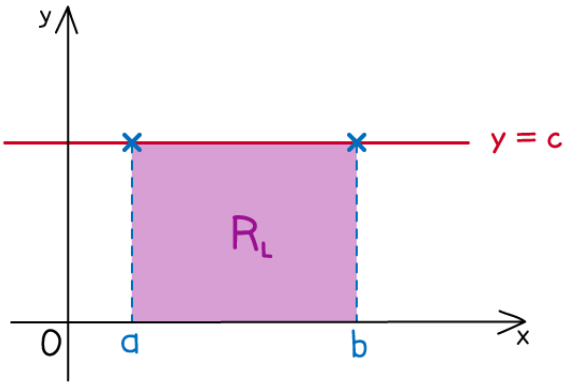
- **STEP 1**
  - Find the intersections of the line and the curve (if not given)
- **STEP 2**
  - Find the area under a curve, $\mathbf{R_C}$, using definite integration
- **STEP 3**
  - Find the area under a line, $\mathbf{R_L}$, either using definite integration **or** the area formulae for basic shapes
- **STEP 4**
  - To find the area, **R**, between the curve and the line **subtract** the **smaller** area **from** the **larger** area
  - If curve on top this will be $\mathbf{R_C - R_L}$
  - If line on top this will be $\mathbf{R_L - R_C}$

- There may be easier ways to find the area under a line

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

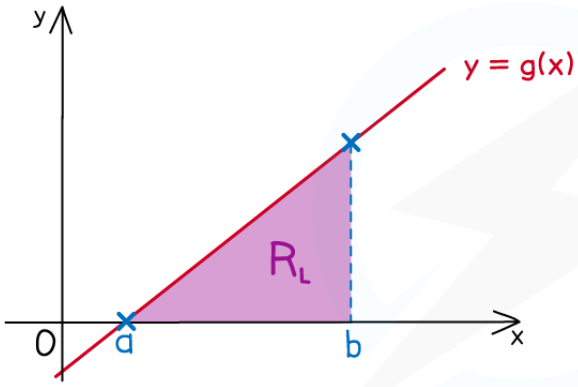
Your notes

AREA UNDER A LINE



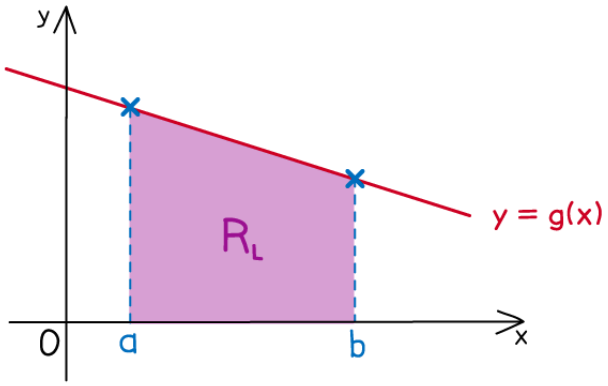
RECTANGLE OR SQUARE

$R_L = c(b - a)$



TRIANGLE

$R_L = \frac{1}{2}(b - a)g(b)$



TRAPEZIUM

$R_L = \frac{b-a}{2}[g(b) + g(a)]$

Copyright © Save My Exams. All Rights Reserved

Can I subtract before integrating?

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- **Yes** - see the next note on "**Area between two curves**"
- Essentially you treat the line as a second curve
  - It can be easier to make mistakes using this method

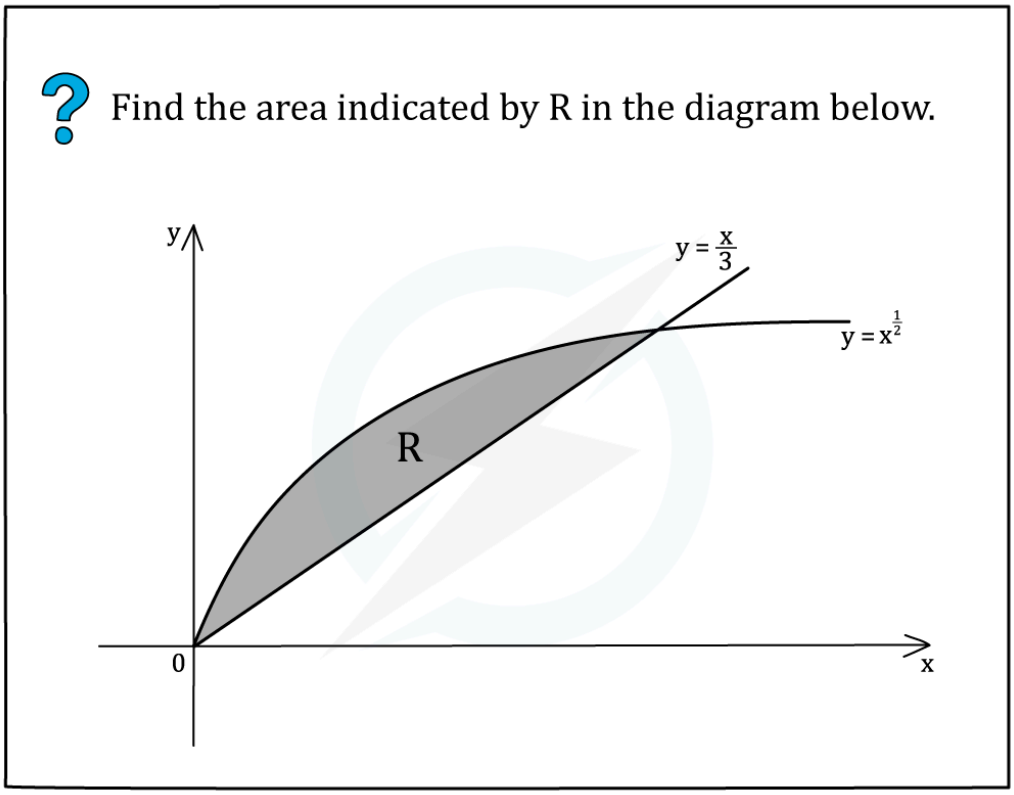

Your notes

Examiner Tip

- Add information to any diagram provided
  - Add axes intercepts, as well as intercepts between lines and curves
- Mark and shade the area you're trying to find
- If no diagram is provided, **sketch** one!

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

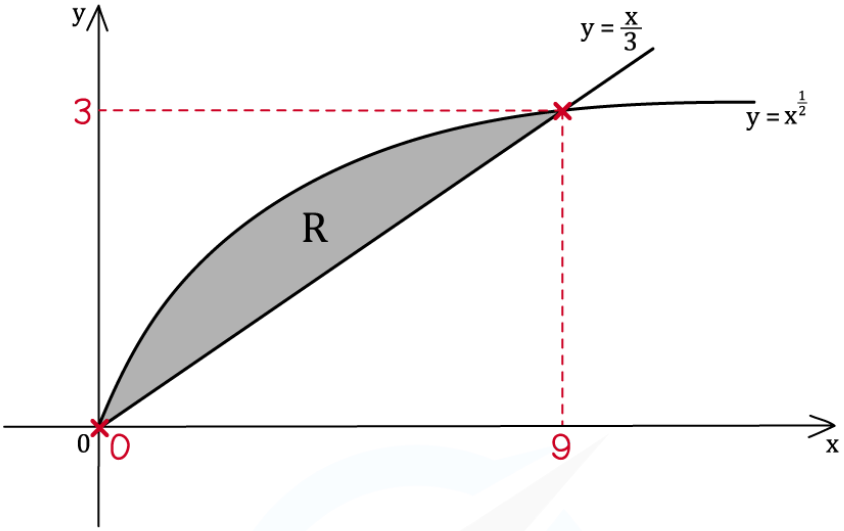


Copyright © Save My Exams. All Rights Reserved
save my exams

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



STEP 1 FIND THE INTERSECTIONS

$\frac{x}{3} = x^{\frac{1}{2}}$ ← SIMULTANEOUS EQUATIONS

$\frac{x^2}{9} = x$

$x^2 = 9x$ ← QUADRATIC

$x^2 - 9x = 0$

$x(x-9) = 0$

$x = 0$ AND $x = 9$ ← ADD TO DIAGRAM

Copyright © Save My Exams. All Rights Reserved
save my exams

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

STEP 2 FIND THE AREA UNDER THE CURVE

$$R_c = \int_0^9 x^{\frac{1}{2}}\,dx = \left[\frac{x^{\frac{3}{2}}}{\frac{3}{2}}\right]_0^9$$

$$= \left[\frac{2}{3}x^{\frac{3}{2}}\right]_0^9$$

$$= \left(\frac{2}{3}(9)^{\frac{3}{2}}\right) - (0)$$

$$R_c = 18$$

STEP 3 FIND THE AREA UNDER THE LINE

$$R_L = \frac{1}{2} \times 9 \times 3$$

TRIANGLE BASE = 9
HEIGHT = 3

$$R_L = \frac{27}{2}$$

Copyright © Save My Exams. All Rights Reserved
save my exams

STEP 4 SUBTRACT SMALL FROM LARGE

$$R = R_c - R_L$$

CURVE IS ON TOP

$$R = 18 - \frac{27}{2}$$

$$R = \frac{9}{2} \text{ SQUARE UNITS}$$

Copyright © Save My Exams. All Rights Reserved
save my exams

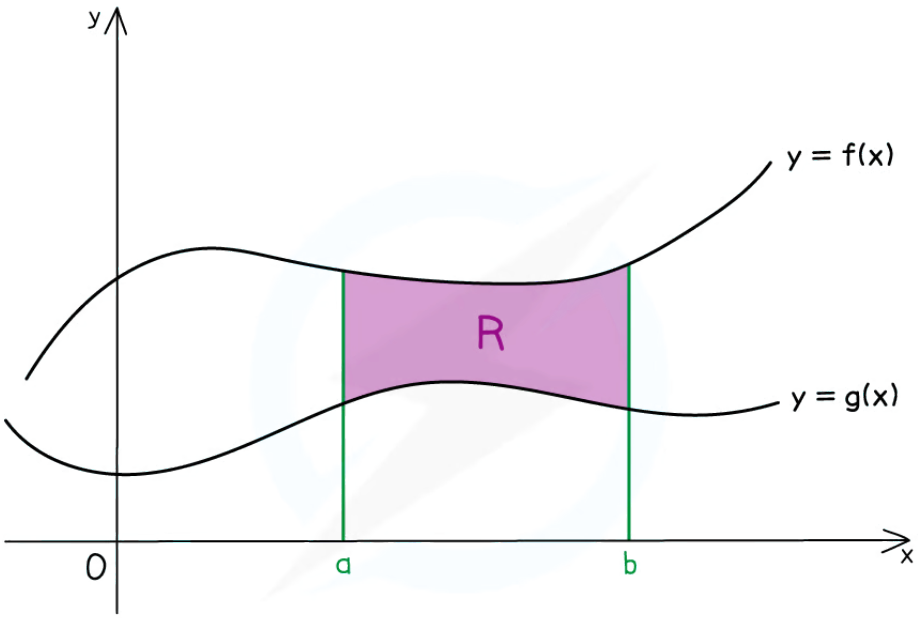
© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Area Between Two Curves

What is the area between two curves?

Your notes

- In general find the definite integral of "upper curve" – "lower curve"
- However this does depend on ...
  - ... the area being found
  - ... if the curves intersect (and cross over)



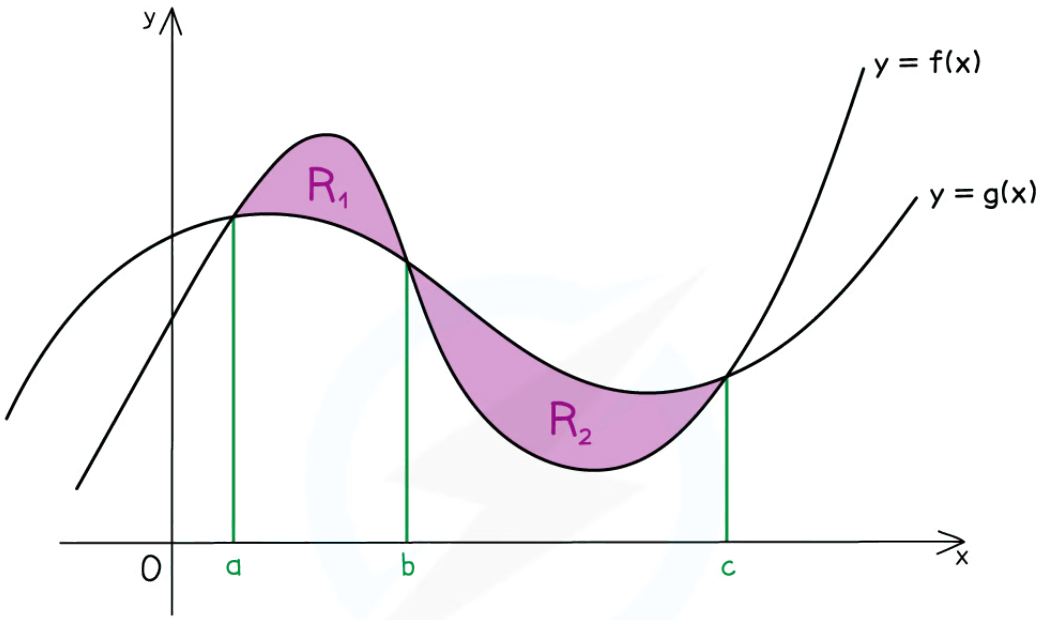
$$R = \int_a^b [f(x) - g(x)]dx$$

Copyright © Save My Exams. All Rights Reserved

- The area may have to be split into separate integrals

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



$$R_1 = \int_a^b [f(x) - g(x)]dx$$

$$R_2 = \int_b^c [g(x) - f(x)]dx$$

Copyright © Save My Exams. All Rights Reserved

- The points at which curves intersect may need to be calculated

How do I find the area between two curves?

- **STEP 1**
  - Find the intersections of the curves if needed

- **STEP 2**
  - Form the integral ...
    - ... using the intersections as limits
    - ... "upper curve" – "lower curve" ...
    - ... and find the value of the integral

- **STEP 3**

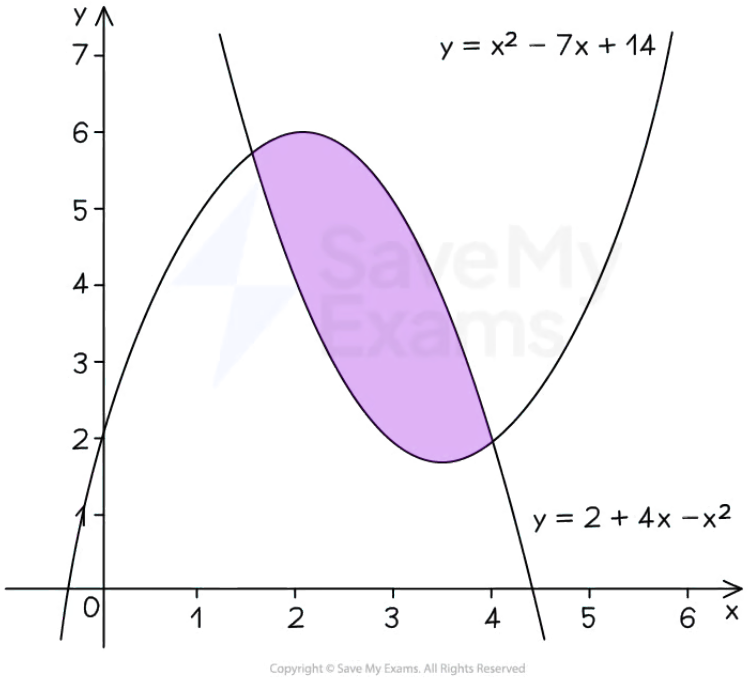
© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- Repeat STEP 2 if more than one area needed

Your notes

- **STEP 4**
  - Add areas together

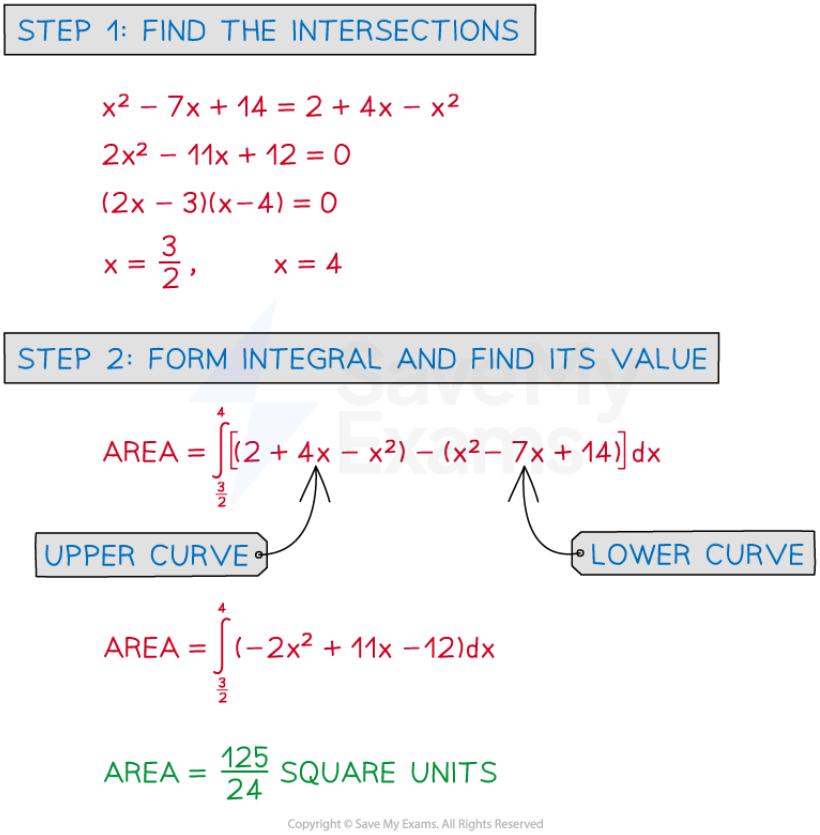
e.g. FIND THE SHADED AREA IN THE DIAGRAM BELOW



Copyright © Save My Exams. All Rights Reserved

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes



Copyright © Save My Exams. All Rights Reserved

Examiner Tip

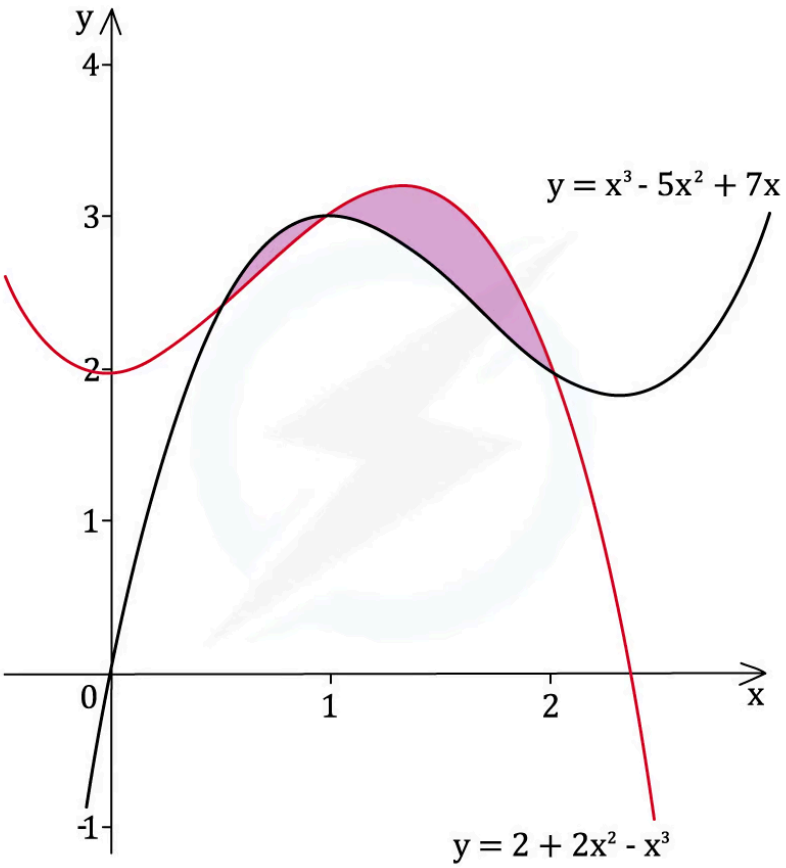
- If no diagram is provided sketch one, even if the curves are not accurate
- Add information to any given diagram as you work through a question

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

Your notes

The diagram below shows a sketch of part of the graphs with equations $y = x^3 - 5x^2 + 7x$ and $y = 2 + 2x^2 - x^3$



Find the shaded area.

Copyright © Save My Exams. All Rights Reserved
save my exams

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

STEP 1: FIND THE INTERSECTIONS

$$x^3 - 5x^2 + 7x = 2 + 2x^2 - x^3$$

$$2x^3 - 7x^2 + 7x - 2 = 0$$

SOLVE USING CALCULATOR

$$x = 2, \quad x = 1, \quad x = \frac{1}{2}$$

STEP 2: FORM AND FIND INTEGRAL FOR FIRST AREA

$$R_1 = \int_{0.5}^{1} \left[(x^3 - 5x^2 + 7x) - (2 + 2x^2 - x^3) \right] dx$$

$$R_1 = \int_{0.5}^{1} (2x^3 - 7x^2 + 7x - 2) dx$$

USE CALCULATOR

$$R_1 = \frac{5}{96}$$

STEP 3: FORM AND FIND INTEGRAL FOR SECOND AREA

$$R_2 = \int_{1}^{2} \left[(2 + 2x^2 - x^3) - (x^3 - 5x^2 + 7x) \right] dx$$

$$R_2 = \int_{1}^{2} (-2x^3 + 7x^2 - 7x + 2) dx$$

$$R_2 = \frac{1}{3}$$

STEP 4: ADD AREAS TOGETHER

$$\text{SHADED AREAS} = R_1 + R_2$$

$$= \frac{5}{96} + \frac{1}{3}$$

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

SHADED AREA = $\frac{37}{96}$ SQUARE UNITS

Copyright © Save My Exams. All Rights Reserved
save my exams

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Integrating Trig Functions

Your notes

Integrating Trig Functions

How do I integrate sin, cos and sec^2?

- The **integrals** for **sine** and **cosine** are

$$\int \sin x \, dx = -\cos x + c$$

$$\int \cos x \, dx = \sin x + c$$

where c is the **constant** of **integration**

- Also, from the **derivative** of $\tan x$

$$\int \sec^2 x \, dx = \tan x + c$$

- For the **linear** function $ax + b$, where a and b are constants,

$$\int \sin (ax + b) \, dx = -\frac{1}{a} \cos (ax + b) + c$$

$$\int \cos (ax + b) \, dx = \frac{1}{a} \sin (ax + b) + c$$

$$\int \sec^2 (ax + b) \, dx = \frac{1}{a} \tan (ax + b) + c$$

- For **calculus** with **trigonometric** functions **angles must** be **measured** in **radians**

Examiner Tip

- Remember to add 'c', the constant of integration, for any indefinite integrals

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Worked example

a) Find, in the form $f(x) + c$, an expression for each integral

i. $\int \cos x \, dx$

ii. $\int \sec^2\left(3x - \frac{\pi}{3}\right) dx$

i. This is a result you should be able to recall from memory.

$$\int \cos x \, dx = \sin x + c$$

ii. Use the standard result: $\int \sec^2(ax + b) \, dx = \frac{1}{a}\tan(ax + b) + c$.

$$\int \sec^2\left(3x - \frac{\pi}{3}\right) dx = \frac{1}{3}\tan\left(3x - \frac{\pi}{3}\right) + c$$

b) A curve has the following equation:

$$y = \int 2\sin\left(2x + \frac{\pi}{6}\right) dx$$

The curve passes through the point with coordinates $\left(\frac{\pi}{3}, \sqrt{3}\right)$.

Find an expression for y.

Factor out the 2 in front of the sin first, and then use the standard result:

$$\int \sin(ax + b) dx = -\frac{1}{a}\cos(ax + b) + c.$$

$$y = 2\int \sin\left(2x + \frac{\pi}{6}\right) dx$$

$$y = 2\left(-\frac{1}{2}\cos\left(2x + \frac{\pi}{6}\right) + c\right)$$

When multiplying the arbitrary constant c, by 2, it is still just an arbitrary constant and so it is still just written as c.

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

$$y = -\cos\left(2x + \frac{\pi}{6}\right) + c$$

Substitute in the given coordinate, and evaluate, to find c.

$$\text{At } x = \frac{\pi}{3},\ y = \sqrt{3}$$

$$\sqrt{3} = -\cos\left(\frac{2\pi}{3} + \frac{\pi}{6}\right) + c$$

$$\sqrt{3} = -\cos\left(\frac{5\pi}{6}\right) + c$$

$$\sqrt{3} = \frac{\sqrt{3}}{2} + c$$

$$c = \frac{\sqrt{3}}{2}$$

Rewrite the full expression for y.

$$y = -\cos\left(2x + \frac{\pi}{6}\right) + \frac{\sqrt{3}}{2}$$

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Integrating e^x & 1/x


Your notes

Integrating e^x & 1/x

How do I integrate exponentials and 1/x?

- The **integrals** involving e^x and $\ln x$ are

$$\int e^x \, dx = e^x + c$$

$$\int \frac{1}{x} \, dx = \ln|x| + c$$

where c is the **constant** of **integration**

- For the **linear** function $(ax + b)$, where a and b are constants,

$$\int e^{ax+b} \, dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \frac{1}{ax+b} \, dx = \frac{1}{a} \ln|ax+b| + c$$

- It follows from the last result that

$$\int \frac{a}{ax+b} \, dx = \ln|ax+b| + c$$

  - which can be deduced using **Reverse Chain Rule**
- With **ln,** it can be useful to write the constant of integration, c, as a logarithm
  - using the laws of logarithms, the answer can be written as a single term
  - $\int \frac{1}{x} \, dx = \ln|x| + \ln k = \ln k|x|$ where k is a constant
  - This is similar to the special case of **differentiating** $\ln(ax + b)$ when $b = 0$

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Worked example

A curve has the gradient function $f'(x) = \frac{3}{3x+2} + e^{4-x}$.

Given the exact value of $f(1)$ is $\ln 10 - e^3$ find an expression for $f(x)$.

To find $f(x)$, we need to integrate the expression for $f'(x)$.

$$f(x) = \int \frac{3}{3x+2} + e^{4-x}\,dx$$

Rewrite as two separate integrals which can be found individually, and factor out any constants.

$$f(x) = 3\int \frac{1}{3x+2}\,dx + \int e^{4-x}\,dx$$

Use the two results:

$$\int \frac{1}{ax+b}\,dx = \frac{1}{a}\ln|ax+b| + c$$

$$\int e^{ax+b}\,dx = \frac{1}{a}e^{ax+b} + c$$

$$f(x) = 3\left(\frac{1}{3}\ln|3x+2|\right) + -1e^{4-x} + c$$

$$f(x) = \ln|3x+2| - e^{4-x} + c$$

The question states that $f(1) = \ln 10 - e^3$, so substitute in $x = 1$ and equate to the given expression.

$$\ln 10 - e^3 = \ln|3(1)+2| - e^{4-1} + c$$

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Simplify and solve to find c.

$$\ln 10 - e^3 = \ln 5 - e^3 + c$$
$$\ln 10 - \ln 5 = c$$

Recall from laws of logarithms that $\ln a - \ln b = \ln \frac{a}{b}$.

$$c = \ln \frac{10}{5} = \ln 2$$

Rewrite the full expression for $f(x)$.

$$f(x) = \ln|3x+2| - e^{4-x} + \ln 2$$

or by combining logs, this could be written as $f(x) = \ln(2|3x+2|) - e^{4-x}$

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Reverse Chain Rule

Your notes

Reverse Chain Rule

What is the reverse chain rule?

- The **Chain Rule** is a way of differentiating two (or more) functions
- The **Reverse Chain Rule** (RCR) refers to **integrating** by **inspection**
  - Spotting that chain rule would be used in the reverse (differentiating) process

How do I know when to use the reverse chain rule?

- The **reverse chain rule** is used when we have the **product** of a **composite function** and the **derivative** of its **second function**
- Integration is trickier than differentiation; many of the shortcuts do not work
  - For example, in general $\int e^{f(x)}\,dx \neq \frac{1}{f'(x)}e^{f(x)}$
  - However, this result **is true** if $f(x)$ is linear $(ax+b)$
- Formally, in **function notation**, the **reverse chain rule** is used for **integrands** of the form

$$I = \int g'(x)f'(g(x))\,dx$$

  - This does not have to be strictly true, but '*algebraically*' it should be
- If the **coefficients** do not match '**adjust** and **compensate**' can be used
  - For example, e^{x^2} differentiates to $2xe^{x^2}$ with the chain rule
    - so $\int 2xe^{x^2}\,dx = e^{x^2} + c$ with the reverse chain rule
  - But to do $\int 5xe^{x^2}dx$ we need to:
    - **Take out** the five: $5\int xe^{x^2}dx$
    - Force a 2 inside (**adjust**) and divide the outside by a 2 (**compensate**): $\frac{5}{2}\int 2xe^{x^2}\,dx$
    - The bit inside the integral is now a reverse chain rule
    - The answer is $\frac{5}{2}e^{x^2} + c$
- A particularly useful instance of the reverse chain rule to recognise is

$$I = \int \frac{f'(x)}{f(x)}\,dx = \ln|f(x)| + c$$

  - i.e. the **numerator** is (almost) the **derivative** of the **denominator**

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

- '**adjust** and **compensate**' may need to be used to deal with any coefficients
  - e.g.

$$I=\int\frac{x^2+1}{x^3+3x}\ \mathrm{d}x=\frac{1}{3}\int 3\frac{x^2+1}{x^3+3x}\ \mathrm{d}x=\frac{1}{3}\int\frac{3x^2+3}{x^3+3x}\ \mathrm{d}x=\frac{1}{3}\ln|x^3+3x|+c$$

Your notes

Examiner Tip

- You can always check your work by differentiating, if you have time

Worked example

A curve has the gradient function $f'(x)=5x^2\sin(2x^3)$.

Given that the curve passes through the point $(0,\ 1)$, find an expression for $f(x)$.

Write f(*x*) as an integral.

$$f(x)=\int 5x^2\sin(2x^3)\,\mathrm{d}x$$

Take 5 out of the integral as a factor.

$$f(x)=5\int x^2\sin(2x^3)\,\mathrm{d}x$$

The main function is sin(...), which would have come from -cos(...).

Adjust and compensate the coefficients.
$2x^3$ would differentiate to $6x^2$ so $-\cos(2x^3)$ would differentiate to $(6x^2)\sin(2\times^3)$

$$f(x)=5\times\left(\frac{1}{6}\right)\times\int(6x^2)\sin(2x^3)\,\mathrm{d}x$$

Integrate.

$$f(x)=\left(\frac{5}{6}\right)\times-\cos(2x^3)+c$$

$$f(x)=-\frac{5}{6}\cos(2x^3)+c$$

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Integrating Composite Functions (ax+b)


Your notes

What is a composite function?

- A **composite function** involves one function being applied after another
- A composite function may be described as a "function of a function"
- This Revision Note focuses on one of the functions being **linear** – i.e. of the form $ax + b$

How do I integrate linear (*ax+b*) functions?

- The reverse chain rule can be used for integrating functions in the form $y = (ax + b)^n$
  - Make sure you are confident using **the chain rule** to **differentiate** functions in the form $y = (ax + b)^n$
  - The reverse chain rule works backwards
- For $n = 2$ you will most likely **expand the brackets** and **integrate each term separately**
- If $n > 2$ this becomes **time-consuming** and if n is not a positive integer we need a different method completely
- To use the **reverse chain rule** $\int (ax+b)^n \, dx$ (provided n is not -1)
  - **Raise** the **power of *n* by 1**
  - **Divide** by this **new power**
  - **Divide** this whole function by the **coefficient of *x***
    - $\int (ax+b)^n \, dx = \frac{(ax+b)^{n+1}}{n+1} \times \frac{1}{a} + c$
- You can **check your answer by differentiating it**
  - You should get the original function when you differentiate your answer
- Note that this method **only works** when the function in the brackets is **linear (*ax* + *b*)**
- The special cases for **trigonometric functions** and **exponential** and **logarithmic functions** are
  - $\int \sin(ax+b) \, dx = -\frac{1}{a}\cos(ax+b) + c$
  - $\int \cos(ax+b) \, dx = \frac{1}{a}\sin(ax+b) + c$
  - $\int e^{ax+b} \, dx = \frac{1}{a}e^{ax+b} + c$
  - $\int \frac{1}{ax+b} \, dx = \frac{1}{a}\ln|ax+b| + c$
- c, in all cases, is the **constant** of **integration**
- All the above can be deduced using **reverse chain rule**
  - However, spotting them can make solutions more efficient

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

Your notes

Worked example

Find the following integrals

a) $\int 3(7-2x)^{\frac{5}{3}}\,\mathrm{d}x$

Name the integral.

$$I = \int 3(7-2x)^{\frac{5}{3}}\,\mathrm{d}x = 3\int (7-2x)^{\frac{5}{3}}\,\mathrm{d}x$$

Using the rule 'raise the power by one, divide by the new power and then multiply by the reciprocal of the derivative' integrate the expression.

$$I = 3\left[\frac{1}{\frac{8}{3}}(7-2x)^{\frac{8}{3}} \times \frac{1}{-2}\right] + c$$

Simplify.

$$I = 3\left[-\frac{3}{16}(7-2x)^{\frac{8}{3}}\right] + c$$

$$\boldsymbol{I = -\frac{9}{16}(7-2x)^{\frac{8}{3}} + c}$$

b) $\int \frac{1}{2}\cos(3x-2)\,\mathrm{d}x$

Name the integral.

$$I = \int \frac{1}{2}\cos(3x-2)\mathrm{d}x = \frac{1}{2}\int \cos(3x-2)\mathrm{d}x$$

Using the rule $\int \cos(ax+b)\,\mathrm{d}x = \frac{1}{a}\sin(ax+b) + c$, integrate the expression.

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers

$$I = \frac{1}{2}\left[\frac{1}{3}\sin(3x-2)\right] + c$$

Simplify.

$$I = \frac{1}{6}\sin(3x-2) + c$$

Your notes

© 2015–2025 Save My Exams, Ltd. · Revision Notes, Topic Questions, Past Papers