CIE A Level Maths: Probability & Statistics 1

Your notes

2.2 Permutations & Combinations

Contents

2.2.1 Arrangements & Factorials

Your notes

Arrangements

How many ways can *n* different objects be arranged?

- When considering how many ways you can arrange a number of **different** objects in a row it's a good idea to think of how many of the objects can go in the first position, how many can go in the second and so on
- For $n = 2$ there are two options for the first position and then there will only be one object left to go in the second position so
  - To arrange the letters A and B we have
    - AB and BA
  - For $n = 3$ there are three options for the first position and then there will be two objects for the second position and one left to go in the third position so
    - To arrange the letters A, B and C we have
      - ABC, ACB, BAC, BCA, CAB and CBA
    - For *n* objects there are n options for the first position, $n - 1$ options for the second position and so on until there is only one object left to go in final position
    - The number of ways of arranging different objects is $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

What happens if the objects are not all different?

- Consider arranging two identical objects, although there are still two different ways you could place the objects down next to each other, the arrangements would look exactly the same
  - To arrange the letters A_1 and A_2 we have
    - $A_1 A_2$ and $A_2 A_1$
    - These are exactly the same, so there is only one way to arrange the letters A and A
  - To arrange the letters A_1, A_2 and C we have
    - $A_1 A_2 C$, $A_2 A_1 C$, $A_1 C A_2$, , $A_2 C A_1$, $C A_1 A_2$, $C A_2 A_1$
    - Although the two letter As were placed separately, they are identical and so each pattern has been repeated twice
    - There are 6 ways to arrange the letters A, A and C, but with some duplicates
    - There are $\frac{6}{2} = 3$ different ways to arrange the letters A, A and C
  - If there are two identical objects within a group of objects to be arranged, the number of ways of arranging different objects should be divided by 2
- Consider arranging three identical objects, although there are still six different ways you could place the objects down next to each other, the arrangements would look exactly the same
  - To arrange the letters A_1, A_2 and A_3
    - $A_1 A_2 A_3$, $A_1 A_3 A_2$, $A_2 A_1 A_3$, $A_2 A_3 A_1$, $A_3 A_1 A_2$, $A_3 A_2 A_1$
    - However, if these were all A, we would have AAA repeated six times
  - To find the number of arrangements of the letters A, A, A and C we would have to consider the number of ways of arranging four letters if they were all different and then divide by the number of

ways AAA is repeated
  - Four different letters could be arranged $4 \times 3 \times 2 \times 1$ = 24 ways
  - AAA would be repeated six times so we would need to divide by 6
  - There are four **different** ways to arrange the letters A, A, A and C
- If there are three identical objects within a group of *n* objects to be arranged, the number of ways of arranging n different objects should be divided by 6
- If there are *r* identical objects within a group of *n* objects to be arranged, the number of ways of arranging n different objects should be divided by the number of ways of arranging *r* different objects
  - If there are *r* identical objects within a group of *n* objects to be arranged, the number of ways of arranging *n* the objects is $n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$ divided by $r \times (r-1) \times (r-2) \times \ldots \times 2 \times 1$

Worked example

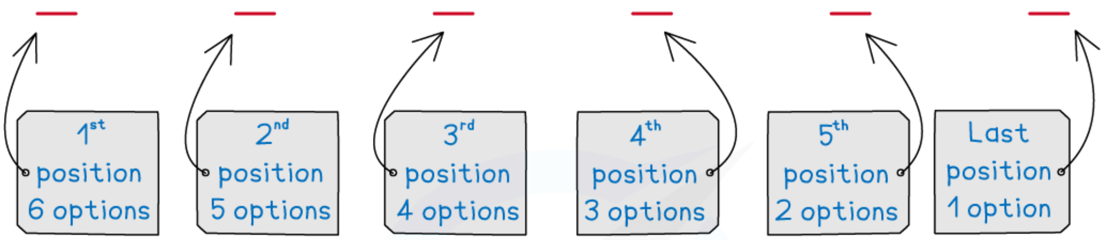
By considering the number of options there are for each letter to go into each position, find how many different arrangements there are of the letters in the word REVISE.

Your notes

First, find the number of ways they could be arranged if they were all different

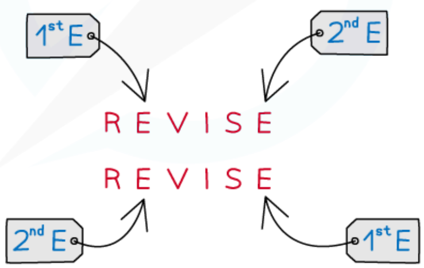
R E V I S E

There are six letters so they can be arranged into six positions:



Then, consider any repeated items:

All arrangements will have E repeated twice, so there will be two of every arrangement



Find the number of ways:

$$\frac{6\times5\times4\times3\times\cancel{2}\times\cancel{1}}{\cancel{2}} = 6\times5\times4\times3 = 360$$

There are 360 arrangements of the word REVISE

Your notes

Factorials

What are factorials?

- Factorials are a type of mathematical operation (just like +, -, ×, ÷)
- The symbol for factorial is !
  - So to take a factorial of any non-negative integer, n , it will be written n! And pronounced 'n *factorial'*
- The factorial function for any integer, n, is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For example, 5 factorial is 5! = 5 × 4 × 3 × 2 × 1
- The factorial of a negative number is not defined
  - You cannot arrange a negative number of items
- 0! = 1
  - There are no positive integers less than zero, so zero items can only be arranged once
- Most normal calculators cannot handle numbers greater than about 70!, experiment with yours to see the greatest value of x such that your calculator can handle $x!$

How are factorials and arrangements linked?

- The number of arrangements of n different objects is $n!$
  - Where $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
- The number of different arrangements of n objects with one object repeated r times and the others all different is $\frac{n!}{r!}$
- The number of different arrangements of n objects with one object repeated r times, another object repeated s times and the other objects all different is $\frac{n!}{r!s!}$

What are the key properties of using factorials?

- Some important relationships to be aware of are:
  - $n! = n \times (n-1)!$
    - Therefore

$$\frac{n!}{(n-1)!} = n$$

  - $n! = n \times (n-1) \times (n-2)!$
    - Therefore

$$\frac{n!}{(n-2)!} = n \times (n-1)$$

- Expressions with factorials in can be simplified by considering which values cancel out in the fraction
  - Dividing a large factorial by a smaller one allows many values to cancel out

$$\frac{7!}{4!}=\frac{7\times6\times5\times4\times3\times2\times1}{4\times3\times2\times1}=7\times6\times5$$

Your notes

Worked example

(i) Show, by writing 8! and 5! in their full form and cancelling, that

$$\frac{8!}{5!}=8\times7\times6$$

(ii) Hence, simplify $\frac{n!}{(n-3)!}$

(iii) The letters A, B, B, B, B, B, C and D are arranged in a row. How many different ways are there to arrange the 8 letters in a row?

Your notes

(i) $5! = 5 \times 4 \times 3 \times 2 \times 1$

$8! = 8 \times 7 \times 6 \times 5 \times 4 \times 3 \times 2 \times 1$

$$\frac{8!}{5!} = \frac{8 \times 7 \times 6 \times \cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}{\cancel{5} \times \cancel{4} \times \cancel{3} \times \cancel{2} \times \cancel{1}}$$

$$\frac{8!}{5!} = 8 \times 7 \times 6$$

(ii) Hence means using the information in the previous question

Let n be 8

n

n − 1

n − 2

$$\frac{8!}{5!} = 8 \times 7 \times 6$$

Then $5 = n - 3$

$$\frac{n!}{(n-3)!} = n \times (n-1) \times (n-2)$$

'OR OTHERWISE' This could also be shown by expanding $n!$ and $(n-3)!$

$$\frac{n!}{(n-3)!} = \frac{n \times (n-1) \times (n-2) \times \cancel{(n-3)} \times \cancel{(n-4)} \times \cancel{\cdots} \times \cancel{2} \times \cancel{1}}{\cancel{(n-3)} \times \cancel{(n-4)} \times \cancel{\cdots} \times \cancel{2} \times \cancel{1}}$$

(iii) A B B B B B C D There are 8 letters but letter B is repeated 5 times

Arranges 8 letters

$$\frac{8!}{5!}$$

There is one letter repeated 5 times

$$\frac{8!}{5!} = 8 \times 7 \times 6 = 336$$

There are 336 arrangements

Examiner Tip

- Arrangements and factorials are tightly interlinked with permutations and combinations
- Make sure you fully understand the concepts in this revision note as they will be fundamental to answering perms and combs exam questions!

Your notes

Your notes

2.2.2 Permutations

Permutations

Are permutations and arrangements the same thing?

- Mathematically speaking yes, a **permutation** is the number of possible arrangements of a set of objects when the **order** of the arrangements matters
- A permutation can either be finding the number of ways to arrange n items or finding the number of ways to arrange r out of n items
- By the reasoning given in the 2.2.1 Arrangements revision note, the number of permutations of n **different** items is $n! = n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$
  - For 5 **different** items there are 5! = 5 × 4 × 3 × 2 = 120 permutations
  - For 6 **different** items there are 6! = 6 × 5 × 4 × 3 × 2 = 720 permutations
  - It is easy to see how quickly the number of possible permutations of different items can increase
  - For 10 different items there are 10! = 3 628 800 possible permutations

How do we handle permutations if there are repeated items?

- Again, by the reasoning given in the 2.2.1 Arrangements revision note, the number of permutations of $\boldsymbol{n}$ **different** items, with **one** of the items repeated $\boldsymbol{r}$ times, is

$$\frac{n!}{r!} = n \times (n-1)) \times \ldots \times (r+1)$$

- The number of permutations of $\boldsymbol{n}$ **different** items, with **one** of the items repeated $\boldsymbol{r}$ times and **another** repeated $\boldsymbol{s}$ times, is $\frac{n!}{r!\,s!}$
- This property will need to be applied to any permutation problem with one or more item(s) repeated a number of times

How do we find r permutations of n items?

- If we only want to find the number of ways to arrange a few out of n different objects, we should consider how many of the objects can go in the first position, how many can go in the second and so on
- If we wanted to arrange 3 out of 5 different objects, then we would have 3 positions to place the objects in, but we would have 5 options for the first position, 4 for the second and 3 for the third
  - This would be 5 × 4 × 3 ways of permutating 3 out of 5 different objects
  - This is equivalent to $\frac{5!}{2!} = \frac{5!}{(5-3)!}$
- If we wanted to arrange 4 out of 10 different objects, then we would have 4 positions to place the objects in, but we would have 10 options for the first position, 9 for the second, 8 for the third and 7 for the fourth
  - This would be 10 × 9 × 8 × 7 ways of permutating 4 out of 10 different objects

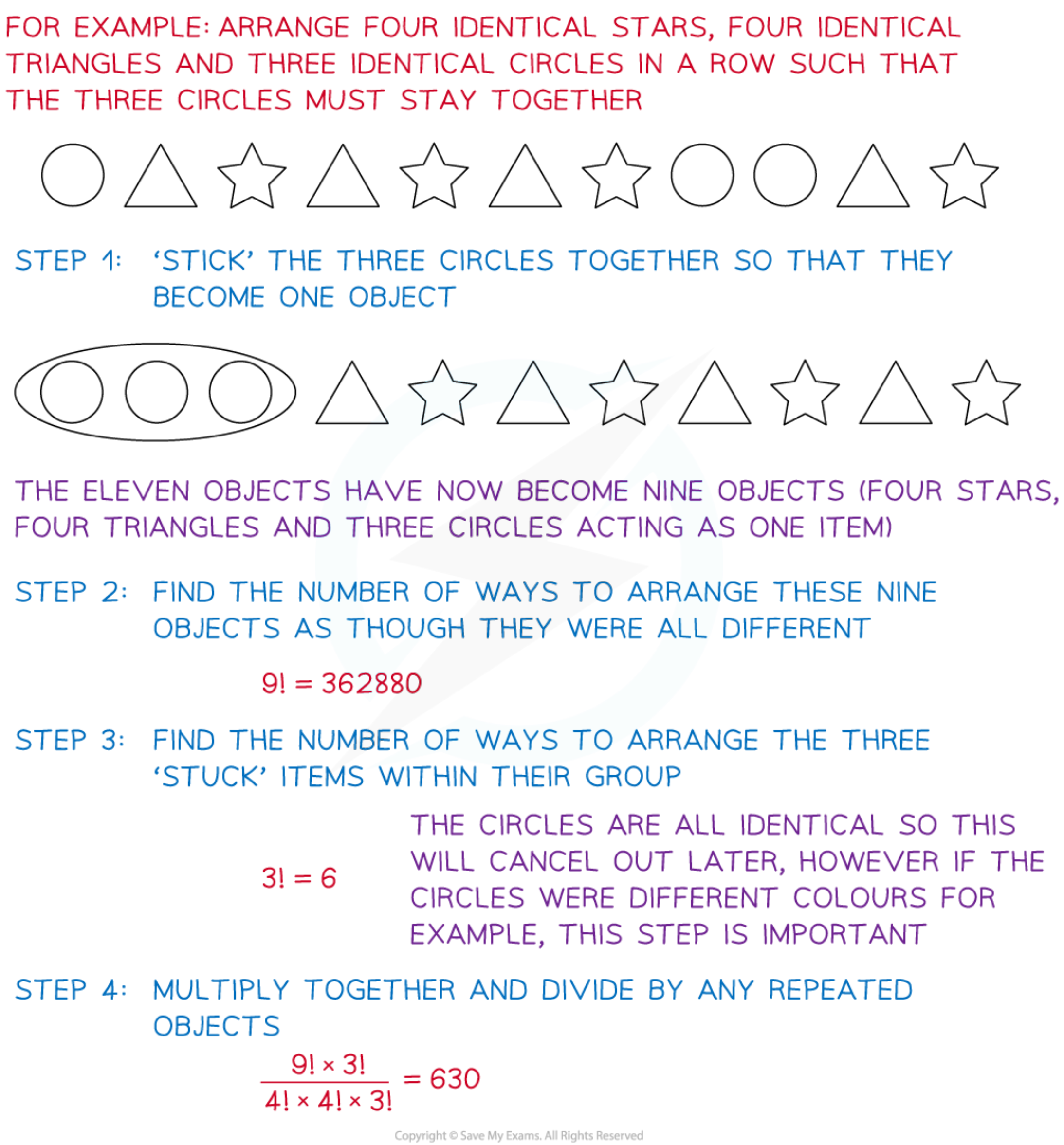
Your notes

- This is equivalent to $\frac{10!}{6!} = \frac{10!}{(10-4)!}$
- If we wanted to arrange *r* out of *n* different objects, then we would have *r* positions to place the objects in, but we would have *n* options for the first position, $(n-1)$ for the second, $(n-2)$ for the third and so on until we reach $(n-(r-1))$
  - This would be $n \times (n-1) \times \ldots \times (n-r+1)$ ways of permutating r out of n different objects
  - This is equivalent to $\frac{n!}{(n-r)!}$
- The function $\frac{n!}{(n-r)!}$ can be written as ${}^{n}\mathrm{P}_{r}$
  - Make sure you can find and use this button on your calculator
- The same function works if we have *n* spaces into which we want to arrange *r* objects, consider
  - for example arranging five people into a row of ten empty chairs

Permutations when two or more items must be together

- If two or more items must stay together within an arrangement, it is easiest to think of these items as 'stuck' together
- These items will become one within the arrangement
- Arrange this 'one' item with the others as normal
- Arrange the items within this 'one' item separately
- Multiply these two arrangements together

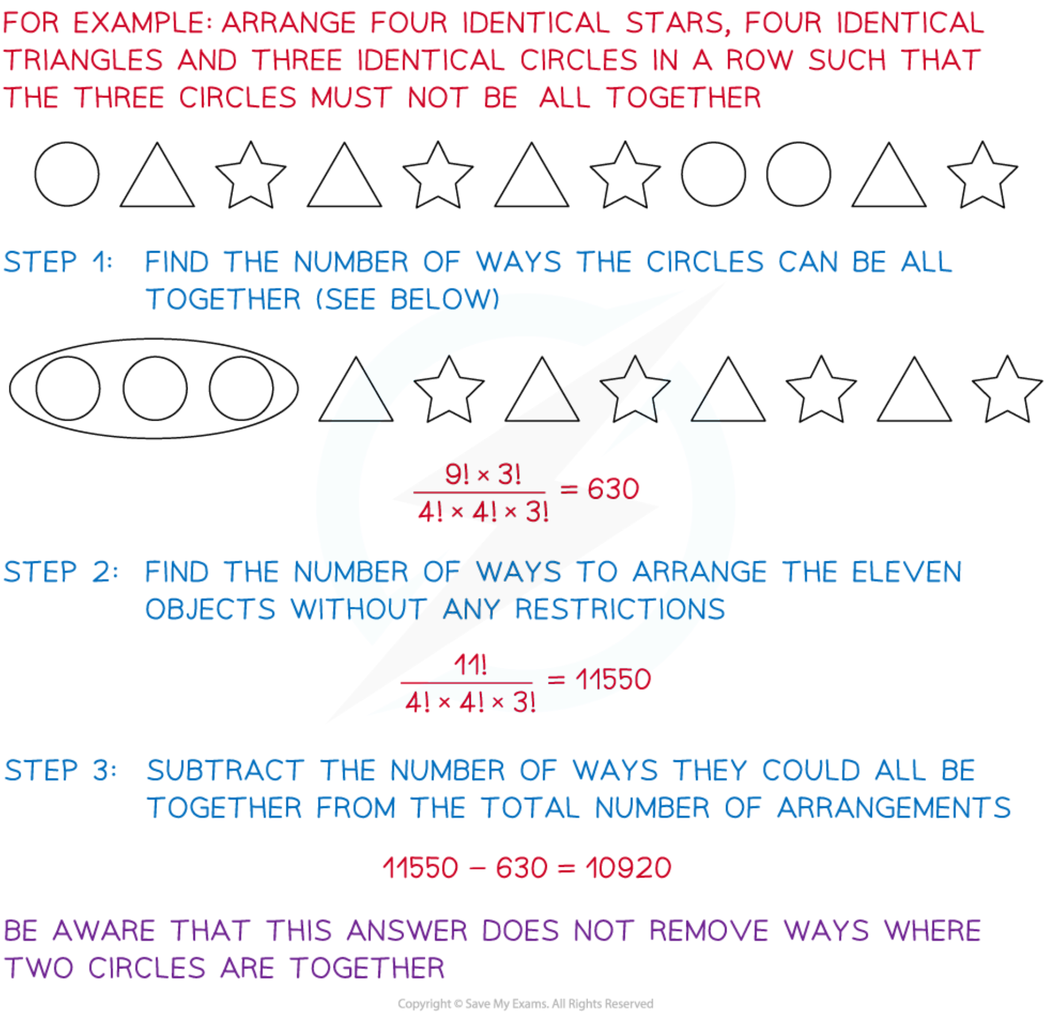
Your notes



Permutations when two or more items cannot be all together

- If **two** items must be **separated** ...
  - consider the number of ways these two items would be together
  - subtract this from the total number of arrangements without restrictions
- If **more than two** items must be separated...
  - consider whether all of them must be **completely separate** (none can be next to each other) or whether they **cannot all be together** (but two could still be next to each other)
  - If they **cannot all be together** then we can treat it the same way as separating two items and subtract the number of ways they would all be together from the total number of permutations of

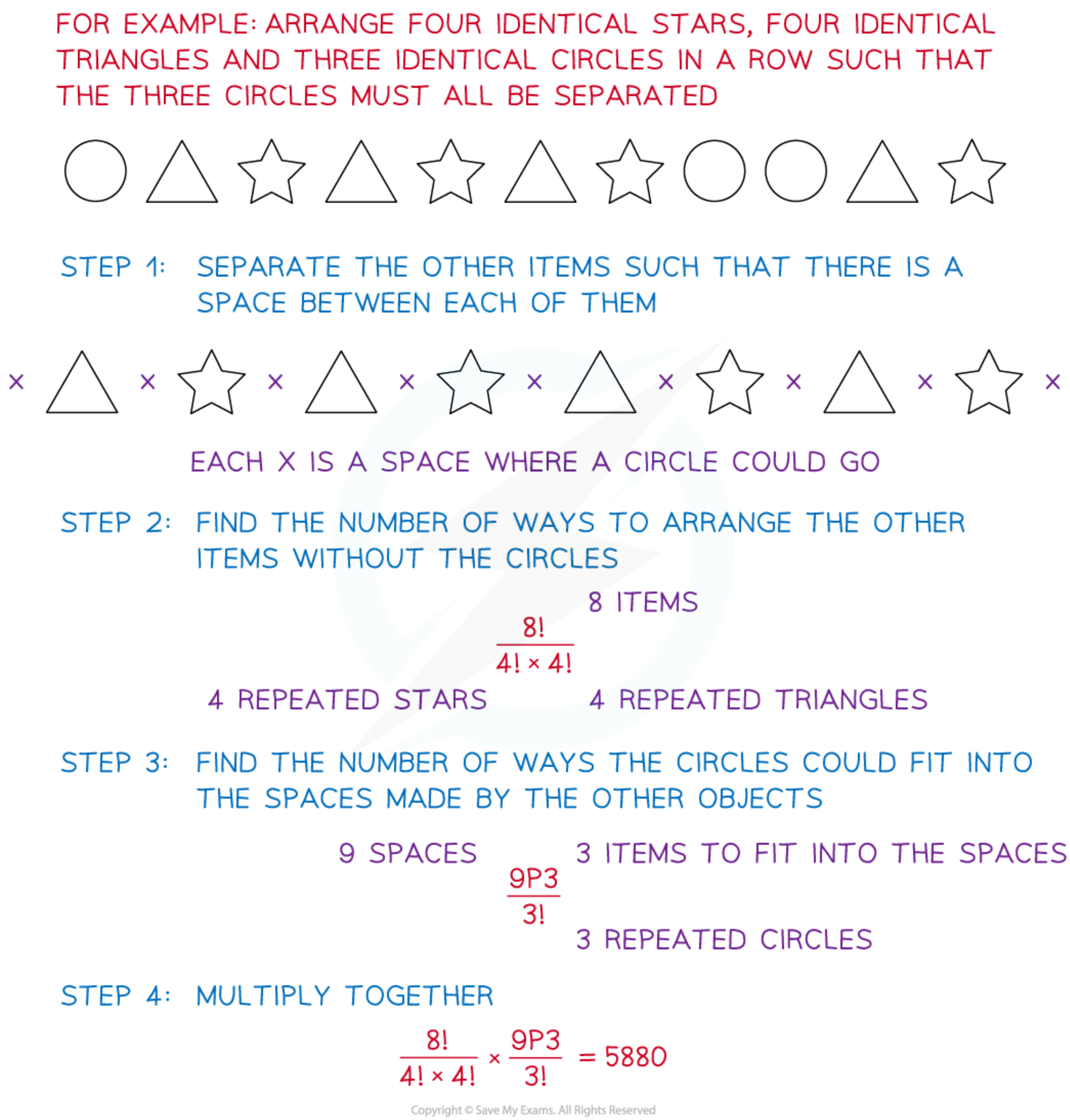
the items, the final answer will include all permutations where two items are still together



Permutations when two or more items must be separated

- If the items must all be **completely separate** then
  - lay out the rest of the items in a line with a space in between each of them where one of the items which cannot be together could go
  - remember that this could also include the space before the first and after the last item
  - You would then be able to fit the items which cannot be together into any of these spaces, using the *r* permutations of *n* items rule $\left({}^{n}\mathrm{P}_{r}\right)$
  - You do not need to fill every space

Your notes



Permutations when two or more items must be in specific places

- Most commonly this would be arranging a word where specific letters would go in the first and last place
- Or arranging objects where specific items have to be at the ends/in the middle
  - Imagine these specific items are stuck in place, then you can find the number of ways to arrange the rest of the items around these 'stuck' items
- Sometimes the items must be grouped
  - For example all vowels must be before the consonants
  - Or all the red objects must be on one side and the blue objects must be on the other

- Find the number of permutations within each group separately and multiply them together
- Be careful to check whether the groups could be in either place
  - e.g. the vowels on one side and consonants on the other
  - or if they must be in specific places (the vowels **before** the consonants)
- If the groups could be in either place than your answer would be multiplied by two
- If there were *n* groups that could be in any order then you're answer would be multiplied by *n*!

Your notes

Worked example

(a) How many ways are there to rearrange the letters in the word BANANAS if the B and the S must be at each end?

(b) How many ways are there to rearrange the letters in the word ORANGES if

(i) the three vowels (A, E and O) must be together?

(ii) the three vowels must NOT all be together?

(iii) the three vowels must all be separated?

(a) How many ways are there to rearrange the letters in the word BANANAS if the B and the S must be at each end?

Your notes

a) B A N A N A S

Three As
Two Ns
One B, one S

Step 1: 'Stick' the 'B' and 'S' at each end

B A A A N N S

Step 2: Find the number of arrangements of the other letters:

Five letters (A A A N N)

$$\frac{5!}{3!2!}$$

Three identical As, two identical Ns

Step 3: Check to see if the 'stuck' items could be swapped around

Yes, be careful here. We have considered the B at the start and the S at the end but these could be swapped around

S A A A N N B

2! ways to arrange the B and S

Step 4: Multiply to get your final answer:

$$\frac{5!}{3!2!} \times 2! = \frac{5!}{3!} = 5 \times 4 = 20$$

Arrangements of the letters in the middle

Arrangements of the outside letters

There are 20 ways

(b) How many ways are there to rearrange the letters in the word ORANGES if

(i) the three vowels (A, E and O) must be together?

(ii) the three vowels must NOT all be together?

(iii) the three vowels must all be separated?

Your notes

b)(i) 'Stick' the three vowels together: O E A R N G S

The number of arrangements of the five items (treating the three vowels as one)

$5! \times 3!$

The number of arrangements of the three vowels within their group

(All the letters are different so no need to divide)

720 ways

(ii) For the three vowels not all together, subtract number of ways they are all together from the total number of arrangements

There are 7 letters with no repeats:

Total number of arrangements $= 7!$

$7! - (5! \times 3!) = 5040 - 720$

4320 ways

(iii) To find the number of ways the three vowels are all separated, first separate out the non-vowels:

There are 4! ways to separate the non-vowels

$\times \; R \times N \times G \times S \; \times$

The blue crosses are spaces where the vowels could go

Multiply the number of permutations of the non-vowels with the number of permutations of the vowels

$4! \times {}^5P_3$

There are 5 spaces for 3 vowels

1440 ways

Your notes

Examiner Tip

- The wording is very important in permutations questions, just one word can change how you answer the question.
- Look out for specific details such as whether three items must all be separated or just cannot be all together (there is a difference).
- Pay attention to whether items must be in alternating order (e.g. red and blue items must alternate, either RBRB... or BRBR...) or whether a particular item must come first (red **then** blue and so on).
- If items should be at the ends, look out for whether they can be at either end or whether one must be at the beginning and the other at the end.

2.2.3 Combinations

Your notes

Combinations

What is the difference between permutations and combinations?

- A **combination** is the number of possible arrangements of a set of objects when the **order** of the arrangements **does not matter**
  - On the other hand a permutation is when the order of arrangement does matter
- A combination will be finding the number of ways to **choose** *r* out of *n* items
  - The order in which the *r* items are chosen is not important
  - For example if we are choosing two letters from the word CAB, AB and BA would be considered the same combination but different permutations

How do we find *r* combinations of *n* items?

- If we want to find the number of ways to **choose** 2 out of 3 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 2 items from 3** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 2 letters from A, B and C
    - There are 6 permutations of 2 letters:

      AB, BA, AC, CA, BC, CB

    - For each combination of 2 letters there are 2 (2 × 1) ways of arranging them (for example, AB and BA)
    - So divide the total number of permutations (6) by the number of ways of arranging each combination (2) to get 3 combinations
- If we want to find the number of ways to **choose** 3 out of 5 different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of 3 items from 5** and then divide by the number of ways of arranging each combination
  - For example if we want to choose 3 letters from A, B, C, D and E

    There are 60 permutations of 3 letters:

    ABC, ACB, BAC, BCA, CAB, CBA, ABD, ADB, etc

    - For each combination of 3 letters there are 6 (3 × 2 ×1) ways of arranging them (for example, ABC, ACB, BAC, BCA, CAB and CBA)
    - So divide the total number of permutations (60) by the number of ways of arranging each combination (6 which is 3!) to get 10 combinations
- If we want to find the number of ways to **choose** *r* items out of *n* different objects, but we don't mind the order in which they are chosen, then we could find the number of **permutations of *r* items from *n*** and then divide by the number of ways of arranging each combination
  - Recall that the formula for *r* permutations of *n* items is ${}^{n}P_{r} = \dfrac{n!}{(n-r)!}$
  - This would include *r* ! ways of repeating each combination

Your notes

- The formula for *r* **combinations** of *n* items is $\frac{{}^{n}P_{r}}{r!} = \frac{n!}{(n-r)!r!}$
- The function $\frac{n!}{(n-r)!r!}$ can be written as ${}^{n}C_{r}$ or $\binom{n}{r}$ and is often read as '*n choose r*'
- Make sure you can find and use the ${}^{n}C_{r}$ button on your calculator
- The formulae for permutations and combinations satisfy the following relationship:

$${}^{n}\mathrm{C}_{r} = \frac{{}^{n}\mathrm{P}_{r}}{r!}$$

What do I need to know about combinations?

- The formula ${}^{n}C_{r} = \frac{n!}{(n-r)!r!}$ is also known as a **binomial** coefficient
- ${}^{n}C_{n} = {}^{n}C_{0} = 1$
  - It is easy to see that there is only one way of arranging *n* objects out of *n* and also there can only be one way of arranging 0 objects out of *n*
  - By considering the formula for this, it reinforces the fact that 0! Must equal 1
- The binomial coefficients are symmetrical, so ${}^{n}C_{r} = {}^{n}C_{n-r}$
  - This can be seen by considering the formula for ${}^{n}C_{r}$
  - ${}^{n}C_{n-r} = \frac{n!}{(n-r)!(n-(n-r))!} = \frac{n!}{r!(n-r)!} = {}^{n}C_{r}$

How do I know when to multiply or add?

- Many questions will ask you to find combinations of a group of different items from a bigger group of a specified number of those different items
  - For example, find the number of ways five questions could be chosen from a bank of twenty different pure and ten different statistics questions
  - The hint in this example is the word 'chosen', this tells you that the order in which the questions are chosen doesn't matter
- Sometimes questions will have restrictions,
  - For example there should be three pure and two statistics chosen from the bank of questions,
  - Or there must be at least two pure questions within the group
- If unsure about whether to add or multiply your options, ask yourself if A **and** B are both needed, or if A **or** B is needed
  - Always **multiply** if the answer is **and,** and **add** if the answer is **or**
  - For example if we needed exactly three pure **and** two statistics questions we would find the amount of each and multiply them

Your notes

- If we could have either five statistics **or** five pure questions we would find them separately and add the answers
- Probabilities can be found with combinations questions by finding the number of options a selection can be made in a particular way and dividing that by the total number of options

How do we handle combinations if some of the objects are identical?

- Sometimes you will be asked to find the number of ways r items can be chosen from n items when some of the items are identical
- You must consider the identical items separately
- For example, if you wanted to choose 4 letters from the word CHOOSE you would have to consider all the options with zero Os, one O and two Os separately

Worked example

Oscar has to choose four books from a reading list to take home over the summer. There are four fantasy books, five historical fiction books and two classics available for him to choose from. In how many ways can Oscar choose four books if he decides to have:

(i) two fantasy books and two historical fictions?

(ii) at least one of each type of book?

(iii) at least two fantasy books?

Your notes

(i) Choosing two fantasy from four: ${}^{4}C_2$

Choosing two historical fiction from five: ${}^{5}C_2$

Total options $= {}^{4}C_2 \times {}^{5}C_2 = 6 \times 10$

We need fantasy **and** historical fiction so **multiply** the options

60 options

(ii) Oscar must have two of one and one of each of the others
Let F represent a fantasy book, H represent historical fiction and C represent a classic

These are the options he can choose **from**

Has	4	5	2	
Can choose	F	H	C	
	1	1	2	${}^{4}C_1 \times {}^{5}C_1 \times {}^{2}C_2 = 4 \times 5 \times 1 = 20$
	1	2	1	${}^{4}C_1 \times {}^{5}C_2 \times {}^{2}C_1 = 4 \times 10 \times 2 = 80$
	2	1	1	${}^{4}C_2 \times {}^{5}C_1 \times {}^{2}C_1 = 6 \times 5 \times 2 = 60$

And

These are the choices he can make

He can choose from one of these options so we add them:

$20 + 80 + 60 = 160$

'Or' so add

160 options

(iii) He could have two, three or four fantasy books:

Always check that these numbers are not bigger than the possible options

Has	4	5	2	
Can choose	F	H	C	
	2	2	0	${}^{4}C_2 \times {}^{5}C_2 \times {}^{2}C_0 = 60$
	2	1	1	${}^{4}C_2 \times {}^{5}C_1 \times {}^{2}C_1 = 60$
	2	0	2	${}^{4}C_2 \times {}^{5}C_0 \times {}^{2}C_2 = 6$
	3	1	0	${}^{4}C_3 \times {}^{5}C_1 \times {}^{2}C_0 = 20$
	3	0	1	${}^{4}C_3 \times {}^{5}C_0 \times {}^{2}C_1 = 8$
	4	0	0	${}^{4}C_4 \times {}^{5}C_0 \times {}^{2}C_0 = 1$

Use a system to make sure you don't forget any
Check that all the options add up to 4

Any one of these could be chosen this is 'OR' so add them

Total number of options $= 60 + 60 + 6 + 20 + 8 + 1 = 155$

155 options

Alternative method: it doesn't matter whether the other books are historical fiction or classics so we could just consider these as 'not fantasy' (N) consider these as 'not fantasy', (N)

4	7	
F	N	
2	2	${}^4C_2 \times {}^7C_2 = 6 \times 21 = 126$
3	1	${}^4C_3 \times {}^7C_1 = 4 \times 7 = 28$
4	0	${}^4C_4 \times {}^7C_0 = 1 \times 1 = 1$

$126 + 28 + 1 = 155$ options

Your notes

Examiner Tip

- It is really important that you can tell whether a question is about permutations or combinations. Look out for key words such as **arrange** (for permutations) or **choose** or **select** (for combinations). Don't be confused if a question asks for the number of **ways**, this could be for either a permutations or a combinations question. Look out for other clues.