IB · DP · Maths

55 mins | 2 questions

Practice Paper 3

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Total Marks /55

1 (a) This question uses De Moivre's theorem to derive an exact form for the value of $\sin\frac{\pi}{5}$.

For a complex number with modulus $r = 1$, De Moivre's theorem is given by

$$(\cos\theta + \text{i}\sin\theta)^n = \cos n\theta + \text{i}\sin n\theta$$

where $n \in \mathbb{Z}^+$ and θ is measured in radians.

Show that the theorem is true for $n = 1$.

(1 mark)

(b) Consider the case when $n = 2$.

(i) Expand $(\cos\theta + \text{i}\sin\theta)^2$.

(ii) By equating real parts from both sides of De Moivre's theorem, show that $\cos^2\theta - \sin^2\theta = \cos 2\theta$.

(iii) By equating imaginary parts, write down an identity for $\sin 2\theta$ in terms of $\sin\theta$ and $\cos\theta$.

(6 marks)

(c) Consider the case when $n = 3$ and let $c = \cos\theta$ and $s = \sin\theta$.

(i) Expand $(c + \mathrm{i}s)^3$.

(ii) By considering real parts, write down an identity for $\cos 3\theta$ in terms of $\sin\theta$ and $\cos\theta$.

(iii) Use the Pythagorean identity $\cos^2\theta + \sin^2\theta = 1$ to rewrite the identity from part (c)(ii) in terms of $\cos\theta$ only, giving your answer in the form

$$p\cos^3\theta - q\cos\theta = \cos 3\theta$$

where p and q are integers to be found.

(7 marks)

(d) The identity for $\sin 5\theta$ is found by equating the imaginary parts of De Moivre's theorem when $n = 5$, then writing the result in terms of $\sin\theta$ only. The identity is given by

$$\sin 5\theta = 16\sin^5\theta - 20\sin^3\theta + 5\sin\theta$$

You may use this identity without proof for the rest of the question.

(i) By substituting $\theta = \frac{\pi}{5}$ into both sides of the identity for $\sin 5\theta$, show that $x = \sin\frac{\pi}{5}$ satisfies the polynomial equation $16x^5 - 20x^3 + 5x = 0$.

(ii) Showing clear algebraic working, solve the polynomial equation in part (d)(i), giving all your solutions as exact values.

(iii) Using a sketch of $y = \sin x$ for $0 < x < \frac{\pi}{2}$, explain why $0 < \sin\frac{\pi}{5} < \frac{\sqrt{2}}{2}$.

(iv) Justifying your choice of solution from part (d)(ii), prove that the exact value of $\sin\frac{\pi}{5}$ is given by

$$\sin\frac{\pi}{5} = \frac{1}{2}\sqrt{\frac{5-\sqrt{5}}{2}}$$

(11 marks)

2 (a) This question explores the sequence of functions $f_n(x) = 1 - x^2 + x^4 .... + (-1)^n x^{2n}$ on the domain $-1 < x < 1$ and uses them to find bounds on the value of π.

Consider the sequence of functions given by

$$f_n(x) = 1 - x^2 + x^4 ... + (-1)^n x^{2n}$$

where $n \in \mathbb{Z}^+$ and $-1 < x < 1$.

The first three functions in the sequence are given below:

$$f_1(x) = 1 - x^2 \qquad f_2(x) = 1 - x^2 + x^4 \qquad f_3(x) = 1 - x^2 + x^4 - x^6$$

(i) Write down the function $f_4(x)$.

(ii) Use your graphic display calculator to explore the stationary points on the graphs of $y = f_n(x)$ over the domain $-1 < x < 1$. Hence copy and complete the following table:

n	Number of local maximum points	Number of local minimum points
1		
2		
3		
4		

(iii) Use your table to predict the numbers of each type of stationary point that will occur on the graphs of $y = f_n(x)$ for all odd values of n.

(iv) Use $f'_2(x)$ to find the exact coordinates of the stationary points for $n = 2$, stating clearly which coordinates correspond to which types of stationary point.

(9 marks)

(b) As $n \to \infty$, the graph of the limit of the sequence of functions $f_n(x)$ is a smooth curve $y = h(x)$ over the domain $-1 < x < 1$.

(i) By considering $f_n(x)$ as $n \to \infty$ as the infinite geometric series

$$1 - x^2 + x^4 - x^6 + \cdots$$

where $-1 < x < 1$, use an appropriate series summation formula to show that

$$h(x) = \frac{1}{1 + x^2}.$$

(ii) Use your graphic display calculator to sketch, on the same set of axes for $-1 < x < 1$, the graphs of $y = f_3(x)$, $y = f_4(x)$ and $y = h(x)$.

(6 marks)

(c) Show that the area under the curve $y = h(x)$ between $x = 0$ and $x = 1$ is equal to $\frac{\pi}{4}$ square units.

(3 marks)

(d) (i) By considering the sketch from part (b)(ii), state with a reason which out of $y = f_3(x)$ or $y = f_4(x)$ would provide an underestimate of the area in part (c) when integrated between 0 and 1, and which would provide an overestimate.

(ii) Calculate the values of $\int_0^1 f_3(x)\ \mathrm{d}x$ and $\int_0^1 f_4(x)\ \mathrm{d}x$, showing your working and giving your answers as exact fractions.

(iii) Hence use the results above to give a lower and upper bound on π, giving your answer in the form $a < \pi < b$, a and b are correct to 2 decimal places.

(8 marks)

(e) Starting from the result that

$$1 - x^2 + x^4 - x^6 + \cdots = \frac{1}{1 + x^2},$$

derive the MacLaurin series for arctanx(as given in the Formula Booklet). You may assume that an infinite series can be integrated term-by-term, although the value of any constant of integration will need to be found.

(4 marks)